

ON ALGEBRA-VALUED R-DIAGONAL ELEMENTS

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ABSTRACT. For an element in an algebra-valued $*$ -noncommutative probability space, equivalent conditions for algebra-valued R-diagonality (a notion introduced by Śniady and Speicher) are proved. Formal power series relations involving the moments and cumulants of such R-diagonal elements are proved. Decompositions of algebra-valued R-diagonal elements into products of the form unitary times self-adjoint are investigated; sufficient conditions, in terms of cumulants, for $*$ -freeness of the unitary and the self-adjoint part are proved, and a tracial example is given where $*$ -freeness fails. The particular case of algebra-valued circular elements is considered; as an application, the polar decomposition of the quasinilpotent DT-operator is described.

1. INTRODUCTION

Let B be a unital algebra over the complex numbers. A B -valued noncommutative probability space is a pair (A, \mathcal{E}) , where A is a unital algebra containing a unitaly embedded copy of B and \mathcal{E} is a conditional expectation, namely, an idempotent linear mapping $\mathcal{E} : A \rightarrow B$ that restricts to the identity on B and satisfies $\mathcal{E}(b_1 a b_2) = b_1 \mathcal{E}(a) b_2$ for every $a \in A$ and $b_1, b_2 \in B$. Elements of A are called random variables or B -valued random variables. When B and A are $*$ -algebras and \mathcal{E} is $*$ -preserving, then the pair (A, \mathcal{E}) is called a B -valued $*$ -noncommutative probability space. The B -valued $*$ -distribution of an element $a \in A$ is, loosely speaking, the collection of $*$ -moments of the form $\mathcal{E}(a^{\epsilon(1)} b_1 a^{\epsilon(2)} \cdots b_{n-1} a^{\epsilon(n)})$ for $n \geq 1$, $\epsilon(1), \dots, \epsilon(n) \in \{1, *\}$ and $b_1, \dots, b_{n-1} \in B$. See Definition 2.5 for a formal version.

We study B -valued R-diagonal random variables in B -valued $*$ -noncommutative probability spaces. Our motivation is the results of [1], where certain random matrices are proved to be asymptotically B -valued R-diagonal. In the scalar-valued case ($B = \mathbf{C}$), R-diagonal random variables were introduced by Nica and Speicher [9] and these include many natural and important examples in free probability theory. In a subsequent paper [7], Nica, Shlyakhtenko and Speicher found several equivalent characterizations of scalar-valued R-diagonal elements. In [12], Śniady and Speicher introduced B -valued R-diagonal elements and proved some equivalent characterizations of them. In this paper we firstly prove some further characterizations of B -valued R-diagonal elements, which are analogous to those of [7]. Secondly, we prove a result involving power series for B -valued R-diagonal elements that is similar to the power series relation between the R-transform and the moment series of a single random variable. Thirdly, we examine polar decompositions of B -valued R-diagonal elements, which is a more delicate topic than in the scalar-valued case. We also study B -valued circular elements (a special case of B -valued R-diagonal elements) and we prove a new result about the polar decomposition of a quasinilpotent DT-operator.

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Definition 1.1. Given $n \in \mathbf{N}$ and $\epsilon = (\epsilon(1), \dots, \epsilon(n)) \in \{1, *\}^n$, we define the *maximal alternating interval partition* $\sigma(\epsilon)$ to be the interval partition of $\{1, \dots, n\}$ whose blocks B are the maximal interval subsets of $\{1, \dots, n\}$ such that if $j \in B$ and $j+1 \in B$, then $\epsilon(j) \neq \epsilon(j+1)$.

The following definition is a reformulation of one of the characterizations of R-diagonality found in [12] and (in the scalar-valued case) in [7].

Definition 1.2. We say that an element a in a B -valued $*$ -noncommutative probability space (A, \mathcal{E}) is *B -valued R-diagonal* if for every integer $k \geq 0$ and every $b_1, \dots, b_{2k} \in B$ we have

$$\mathcal{E}(ab_1a^*b_2ab_3a^* \cdots b_{2k-2}ab_{2k-1}a^*b_{2k}a) = 0,$$

(namely, odd alternating moments vanish) and, for every integer $n \geq 1$, every $\epsilon \in \{1, *\}^n$ and every choice of $b_1, b_2, \dots, b_n \in B$, we have

$$\mathcal{E} \left(\prod_{B \in \sigma(\epsilon)} \left(\left(\prod_{j \in B} a^{\epsilon(j)} b_j \right) - \mathcal{E} \left(\prod_{j \in B} a^{\epsilon(j)} b_j \right) \right) \right) = 0, \quad (1)$$

where in each of the three products above, the terms are taken in order of increasing indices.

Remark 1.3. Clearly a is B -valued R-diagonal if and only if a^* is B -valued R-diagonal.

Remark 1.4. It is not difficult to see, by expansion of the left-hand-side of (1) and induction on n , that the B -valued $*$ -distribution of a B -valued R-diagonal element is completely determined by the collection of even, alternating moments, namely, those of the form

$$\mathcal{E}(a^*b_1ab_2a^*b_3a \cdots b_{2k-2}a^*b_{2k-1}a) \quad \text{and} \quad \mathcal{E}(ab_1a^*b_2ab_3a^* \cdots b_{2k-2}ab_{2k-1}a^*).$$

In Theorem 3.1, we prove equivalence of seven conditions for a B -valued random variable a , including the condition for R-diagonality in Definition 1.2. (For some of these equivalences we refer to [12].) The six others are, in fact, B -valued analogues of those found in [7] for the case $B = \mathbf{C}$. One of these, condition (g), is in terms of Speicher's noncrossing cumulants for the pair (a, a^*) , namely, that only those associated with even alternating sequences in a and a^* may be nonvanishing. Another, condition (f) of Theorem 3.1, is that the matrix $\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ is free from $M_2(B)$ with amalgamation over the diagonal matrices with entries in B , while still another, condition (b), is an easy reformulation of Definition 1.2. The proofs of the various equivalences are similar to those found in [7].

A unitary element of a B -valued $*$ -noncommutative probability space (A, \mathcal{E}) is, of course, an element $u \in A$ such that $u^*u = uu^* = 1$. A *Haar unitary element* is a unitary element satisfying $\mathcal{E}(u^k) = 0$ for all integers $k \geq 1$. In the tracial, scalar-valued case, it is well known (see Proposition 2.6 of [7]) that being R-diagonal is equivalent to having the same $*$ -distribution as an element of the form up , where u is Haar unitary and p is self-adjoint and such that $\{u, u^*\}$ and $\{p\}$ are free. (In the case of a \mathbf{C}^* -noncommutative probability space, one can also take $p \geq 0$ — this is well known, or see, for example, Corollary 5.9 for a proof.) The analogous statement is not true in the tracial, algebra-valued case. Example 6.9 provides a counter example. However, in Proposition 5.5 and Theorem 5.8, we do characterize, in terms of cumulants, when an algebra-valued R-diagonal element has the same B -valued $*$ -distribution as a product up with p self-adjoint, with u a B -normalizing Haar unitary element and with $\{u, u^*\}$ and $\{p\}$ free over B . As an application, Corollary 6.8 shows that the polar decomposition of the quasinilpotent DT-operator has this form.

The contents of the rest of the paper are as follows: Section 2 briefly recalls the formulation from [8] of Speicher's B -valued cumulants, introduces notation and proves some

straightforward results about traces and about self-adjointness of B -valued distributions and cumulants. Section 3 deals with the equivalence of the seven conditions that characterize algebra-valued R-diagonal elements. Section 4 establishes formal power series relations involving B -valued alternating moments and B -valued alternating cumulants of an R-diagonal element. Section 5 proves conditions for traciality of $*$ -distributions of algebra-valued R-diagonal elements and examines polar decompositions and the like for R-diagonal elements. Section 6 examines algebra-valued circular elements, which comprise a special case of algebra-valued R-diagonal elements; the notion of these has appeared before, notably in work of Śniady [11]. This section also contains Example 6.9 concerning the polar decomposition, mentioned above. Finally, in an appendix, we investigate the distribution (with respect to a trace) of the positive part of the operator in this example.

2. CUMULANTS, TRACES, AND $*$ -DISTRIBUTIONS

In this section, we briefly recall a formulation (from [8]) of Speicher's theory [13] of B -valued cumulants and describe the notation we will use. We also prove some straightforward results about traciality and self-adjointness.

Given a family $(a_i)_{i \in I}$ of random variables in a B -valued noncommutative probability space (A, \mathcal{E}) , and given $j = (j(1), \dots, j(n)) \in \bigcup_{n \geq 1} I^n$, the corresponding *cumulant map* is a \mathbf{C} -multilinear map $\alpha_j : B^{n-1} \rightarrow B$. These are defined, recursively, by the moment-cumulant formula

$$\mathcal{E}(a_{j(1)} b_1 a_{j(2)} \cdots b_{n-1} a_{j(n)}) = \sum_{\pi \in \text{NC}(n)} \hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}], \quad (2)$$

where $\text{NC}(n)$ is the set of all noncrossing partitions of $\{1, \dots, n\}$ and where for $\pi \in \text{NC}(n)$, $\hat{\alpha}_j(\pi)$ is a multilinear map defined in terms of the cumulant maps $\alpha_{j'}$ for the j' obtained by restricting j to the blocks of π . In detail, $\hat{\alpha}_j(\pi)$ can be specified (recursively) as follows. If $\pi = 1_n$, then

$$\hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}] = \alpha_j(b_1, \dots, b_{n-1}),$$

while if $\pi \neq 1_n$ then, selecting an interval block $\{p, p+1, \dots, p+q-1\} \in \pi$ with $p \geq 1$ and $q \geq 1$, letting $\pi' \in \text{NC}(n-q)$ be obtained by restricting π to $\{1, \dots, p-1\} \cup \{p+q, \dots, n\}$ and renumbering to preserve order, and letting $j' \in I^{n-q}$ and $j'' \in I^q$ be

$$j' = (j(1), j(2), \dots, j(p-1), j(p+q), j(p+q+1), \dots, j(n)), \quad (3)$$

$$j'' = (j(p), j(p+1), \dots, j(p+q-1)), \quad (4)$$

we have

$$\begin{aligned} \hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}] &= \\ &= \begin{cases} \hat{\alpha}_{j'}(\pi')[b_1, \dots, b_{p-2}, \\ \quad b_{p-1} \alpha_{j''}(b_p, \dots, b_{p+q-2}) b_{p+q-1}, \\ \quad \quad \quad b_{p+q}, \dots, b_{n-1}], & p \geq 2, p+q-1 < n \\ \hat{\alpha}_{j'}(\pi')[b_1, \dots, b_{p-2}] b_{p-1} \alpha_{j''}(b_p, \dots, b_{n-1}), & p \geq 2, p+q-1 = n, \\ \alpha_{j''}(b_1, \dots, b_{q-1}) b_q \hat{\alpha}_{j'}(\pi')[b_{q+1}, \dots, b_{n-1}], & p = 1, q < n. \end{cases} \end{aligned} \quad (5)$$

We will use the notation ψ_j for the multilinear moment map

$$\psi_j(b_1, \dots, b_{n-1}) = \mathcal{E}(a_{j(1)} b_1 a_{j(2)} \cdots b_{n-1} a_{j(n)}). \quad (6)$$

Given a tracial linear functional on B and a B -valued noncommutative probability space (A, \mathcal{E}) , we will now characterize, in terms of B -valued cumulants, when the composition $\tau \circ \mathcal{E}$ is tracial.

In the following lemma and its proof, we will use the notation c for cyclic left permutations. In particular,

$$\text{if } j = (j(1), \dots, j(n)) \in I^n \text{ then } c(j) = (j(2), j(3), \dots, j(n), j(1)),$$

and if $\pi \in \text{NC}(n)$ for some n , then $c(\pi) \in \text{NC}(n)$ is the permutation obtained from π by applying the mapping

$$j \mapsto \begin{cases} n, & j = 1 \\ j - 1, & j > 1 \end{cases}$$

to the underlying set $\{1, \dots, n\}$; so, for example, if $\pi = \{\{1, 2\}, \{3, 4, 5\}\} \in \text{NC}(5)$, then $c(\pi) = \{\{1, 5\}, \{2, 3, 4\}\}$.

Lemma 2.1. *Let Θ be the B -valued distribution of a family $(a_i)_{i \in I}$ with corresponding moment maps ψ_j and cumulant maps α_j . Suppose τ is a tracial linear functional on B . Fix $n \geq 2$ and suppose that for all $m \in \{1, 2, \dots, n-1\}$, all $j' \in I^m$ and all $b_1, \dots, b_m \in B$, we have*

$$\tau(\alpha_{j'}(b_1, \dots, b_{m-1})b_m) = \tau(b_1\alpha_{c(j')}(b_2, \dots, b_m)). \quad (7)$$

Then for every $\pi \in \text{NC}(n) \setminus \{1_n\}$, $j \in I^n$ and $b_1, \dots, b_n \in B$, we have

$$\tau(\hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}]b_n) = \tau(\hat{\alpha}_{c(j)}(c(\pi))[b_2, \dots, b_n]b_1). \quad (8)$$

Proof. We proceed by induction on n . To begin, if $n = 2$, then $\pi = \{\{1\}, \{2\}\}$ and using (5) we have

$$\tau(\hat{\alpha}_j(\pi)[b_1]b_2) = \tau(\alpha_{(j(1))}b_1\alpha_{(j(2))}b_2) = \tau(b_1\alpha_{(j(2))}b_2\alpha_{(j(1))}) = \tau(b_1\hat{\alpha}_{c(j)}(\pi)[b_2]),$$

as required. Assume $n \geq 3$. By the induction hypothesis, for every $m \in \{1, \dots, n-1\}$, $j' \in I^m$ and $\pi' \in \text{NC}(m)$, including, by the original hypothesis (7), the case $\pi' = 1_m$, we have

$$\tau(\hat{\alpha}_{j'}(\pi')[b_1, \dots, b_{m-1}]b_m) = \tau(\hat{\alpha}_{c(j')}(c(\pi'))[b_2, \dots, b_m]b_1). \quad (9)$$

Since $\pi \neq 1_n$, there is an interval block $\{p, p+1, \dots, p+q-1\} \in \pi$ with $p \geq 2$ and $q \geq 1$. Let j' and j'' be as in (3)–(4) and let $\pi' \in \text{NC}(n-q)$ be as described above those equations. If $p+q-1 < n$ and $p \geq 3$, then

$$\begin{aligned} \tau(\hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}]b_n) &= \tau(\hat{\alpha}_{j'}(\pi')[b_1, \dots, b_{p-2}, b_{p-1}\alpha_{j''}(b_p, \dots, b_{p+q-2})b_{p+q-1}, b_{p+q}, \dots, b_{n-1}]b_n) \\ &= \tau(\hat{\alpha}_{c(j')}(c(\pi'))[b_2, \dots, b_{p-2}, b_{p-1}\alpha_{j''}(b_p, \dots, b_{p+q-2})b_{p+q-1}, b_{p+q}, \dots, b_n]b_1) \\ &= \tau(\hat{\alpha}_{c(j)}(c(\pi))[b_2, \dots, b_n]b_1), \end{aligned}$$

where we have used, respectively, the first case on the right-hand-side of (5), (9) and again the first case on the right-hand-side of (5). If $p+q-1 < n$ and $p = 2$, then

$$\begin{aligned} \tau(\hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}]b_n) &= \tau(\hat{\alpha}_{j'}(\pi')[b_1\alpha_{j''}(b_2, \dots, b_q)b_{q+1}, b_{q+2}, \dots, b_{n-1}]b_n) \\ &= \tau(\hat{\alpha}_{c(j')}(c(\pi'))[b_{q+2}, \dots, b_n]b_1\alpha_{j''}(b_2, \dots, b_q)b_{q+1}) \\ &= \tau(\alpha_{j''}(b_2, \dots, b_q)b_{q+1}\hat{\alpha}_{c(j')}(c(\pi'))[b_{q+2}, \dots, b_n]b_1) \\ &= \tau(\hat{\alpha}_{c(j)}(c(\pi))[b_2, \dots, b_n]b_1), \end{aligned}$$

where we have used, respectively, the first case on the right-hand-side of (5), (9), the trace property of τ and the third case on the right-hand-side of (5). The case of $p \geq 3$ and

$p + q - 1 = n$ is done similarly, while if $p = 2$ and $q = n - 1$, then

$$\begin{aligned}\tau(\hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}]b_n) &= \tau(\alpha_{j(1)}b_1\alpha_{j''}(b_2, \dots, b_{n-1})b_n) \\ &= \tau(\alpha_{j''}(b_2, \dots, b_{n-1})b_n\alpha_{j(1)}b_1) \\ &= \tau(\hat{\alpha}_{c(j)}(c(\pi))[b_2, \dots, b_n]b_1),\end{aligned}$$

where we have used, respectively, the second case on the right-hand-side of (5), the trace property of τ and the third case on the right-hand-side of (5). Thus, (8) holds and the lemma is proved. \square

Proposition 2.2. *Let Θ be the B -valued distribution of a family $(a_i)_{i \in I}$ with corresponding moment maps ψ_j and cumulant maps α_j . Suppose τ is a tracial linear functional on B . Then the following are equivalent:*

- (i) *the linear functional $\tau \circ \Theta$ on $B\langle X_i \mid i \in I \rangle$ is tracial,*
- (ii) *$\forall n \geq 2 \forall j \in I^n \forall b_1, \dots, b_n \in B$, we have*

$$\tau(\psi_j(b_1, \dots, b_{n-1})b_n) = \tau(b_1\psi_{c(j)}(b_2, \dots, b_n)), \quad (10)$$

- (iii) *$\forall n \geq 1 \forall j \in I^n \forall b_1, \dots, b_n \in B$, we have*

$$\tau(\alpha_j(b_1, \dots, b_{n-1})b_n) = \tau(b_1\alpha_{c(j)}(b_2, \dots, b_n)). \quad (11)$$

Proof. The equivalence of (i) and (ii) is easily seen from the definition (6) of ψ_j .

The proof of (iii) \implies (ii) follows from the moment-cumulant formula (2) and Lemma 2.1.

We will prove (ii) \implies (iii) using the moment-cumulant formula (2) and Lemma 2.1. Suppose (ii) holds and let us show (11) holds by induction on $n \geq 1$. The case $n = 1$ is from the tracial property of τ . Fix $n_0 \geq 2$ and suppose (11) holds for all $n < n_0$. Then, by Lemma 2.1, for all $\pi \in \text{NC}(n_0) \setminus \{1_{n_0}\}$, all $j \in I^{n_0}$ and all $b_1, \dots, b_{n_0} \in B$, we have

$$\tau(\hat{\alpha}_j(\pi)[b_1, \dots, b_{n_0-1}]b_{n_0}) = \tau(\hat{\alpha}_{c(j)}(c(\pi))[b_2, \dots, b_{n_0}]b_1).$$

Combining this with the moment-cumulant formula (2) and using (10), we get

$$\begin{aligned}\tau(\alpha_j(b_1, \dots, b_{n_0-1})b_{n_0}) &= \tau(\psi_j(b_1, \dots, b_{n_0-1})b_{n_0}) - \sum_{\pi \in \text{NC}(n_0) \setminus \{1_{n_0}\}} \tau(\hat{\alpha}_j(\pi)[b_1, \dots, b_{n_0-1}]b_{n_0}) \\ &= \tau(\psi_{c(j)}(b_2, \dots, b_{n_0})b_1) - \sum_{\pi \in \text{NC}(n_0) \setminus \{1_{n_0}\}} \tau(\hat{\alpha}_{c(j)}(c(\pi))[b_2, \dots, b_{n_0}]b_1) \\ &= \tau(\alpha_{c(j)}(b_2, \dots, b_{n_0})b_1),\end{aligned}$$

as required. \square

We now turn to questions of self-adjointness. Suppose B is a $*$ -algebra and consider a family $(a_i)_{i \in I}$ of B -valued random variables in a B -valued noncommutative probability space (A, \mathcal{E}) . Consider an involution $s : I \rightarrow I$. Let $B\langle X_i \mid i \in I \rangle$ be endowed with the $*$ -algebra structure coming from the $*$ -operation on B and by setting $X_i^* = X_{s(i)}$ for all $i \in I$. For each $n \geq 1$, let $\tilde{s} : I^n \rightarrow I^n$ be defined by

$$\tilde{s}((j(1), j(2), \dots, j(n))) = (s(j(n)), \dots, s(j(2)), s(j(1))).$$

Lemma 2.3. *Let α_j be the cumulant maps of the family $(a_i)_{i \in I}$. Fix $n \geq 2$ and suppose that for all $m \in \{1, 2, \dots, n-1\}$, all $j' \in I^m$ and all $b_1, \dots, b_{m-1} \in B$, we have*

$$\alpha_{j'}(b_1, \dots, b_{m-1})^* = \alpha_{\tilde{s}(j')}(b_{m-1}^*, \dots, b_1^*).$$

Then for all $\pi \in \text{NC}(n) \setminus \{1_n\}$, all $j \in I^n$ and all $b_1, \dots, b_{n-1} \in B$, we have

$$\hat{\alpha}_j(\pi)[b_1, \dots, b_{n-1}]^* = \alpha_j(r(\pi))[b_{n-1}^*, \dots, b_1^*],$$

where $r(\pi)$ is the noncrossing partition obtained from π by applying the reflection $j \mapsto n - j$ to all elements in the underlying set $\{1, \dots, n - 1\}$.

Proof. This follows in a straightforward manner by induction on the number of blocks in π , from the recursion formula (5) for cumulants. \square

The following proposition gives a criterion for self-adjointness of the distribution of a family of B -valued random variables in terms of properties of the B -valued cumulant maps.

Proposition 2.4. *The following are equivalent:*

- (i) *the linear mapping $\Theta : B\langle X_i \mid i \in I \rangle \rightarrow B$ is self-adjoint.*
- (ii) *$\forall n \geq 1 \forall j \in I^n \forall b_1, \dots, b_{n-1} \in B$, we have*

$$\psi_j(b_1, \dots, b_{n-1})^* = \psi_{\bar{s}(j)}(b_{n-1}^*, \dots, b_1^*),$$

- (iii) *$\forall n \geq 1 \forall j \in I^n \forall b_1, \dots, b_n \in B$, we have*

$$\alpha_j(b_1, \dots, b_{n-1})^* = \alpha_{\bar{s}(j)}(b_{n-1}^*, \dots, b_1^*).$$

Moreover, if (A, \mathcal{E}) is a $*$ -noncommutative probability space and if $a_i^* = a_{s(i)}$ for all $i \in I$, then the above conditions are satisfied.

Definition 2.5. If the conditions in the last sentence of Proposition 2.4 hold, then we say Θ is the B -valued $*$ -distribution of the family $(a_i)_{i \in I}$.

Proof of Proposition 2.4. The equivalence of (i) and (ii) follows directly from the definitions. The equivalence of (ii) and (iii) is a straightforward application of the moment-cumulant formula and Lemma 2.3. \square

We now suppose that B is a C^* -algebra and Θ is a self-adjoint B -valued distribution as in (i) of Proposition 2.4. We say that Θ is *positive* if $\Theta(p^*p) \geq 0$ for every $p \in B\langle X_i \mid i \in I \rangle$. It can be difficult to verify positivity of a $*$ -distribution Θ only from knowing the cumulants, though there are special cases that are exceptions to this statement (for example, the B -valued semicircular and B -valued circular elements, discussed in Section 6).

3. ALGEBRA-VALUED R-DIAGONAL ELEMENTS

Let B be a unital $*$ -algebra and let (A, \mathcal{E}) be a B -valued $*$ -noncommutative probability space.

It is convenient to introduce here some notation we will use below. By an *enlargement* of (A, \mathcal{E}) , we will mean a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ with an embedding $A \hookrightarrow \tilde{A}$ so that the diagram

$$\begin{array}{ccc} A & \hookrightarrow & \tilde{A} \\ \cup & & \cup \\ B & = & B \end{array}$$

commutes and $\tilde{\mathcal{E}}|_A = \mathcal{E}$.

For an element $a \in A$, consider the following sets of words and their centerings, formed from alternating a and a^* , with elements of B between:

$$\mathcal{P}_{11} = \{b_0 a b_1 a^* b_2 a b_3 a^* b_4 \cdots a b_{2k-1} a^* b_{2k} a b_{2k+1} \mid k \geq 0, b_0, \dots, b_{2k+1} \in B\} \quad (12)$$

$$\mathcal{P}_{22} = \{b_0 a^* b_1 a b_2 a^* b_3 a b_4 \cdots a^* b_{2k-1} a b_{2k} a^* b_{2k+1} \mid k \geq 0, b_0, \dots, b_{2k+1} \in B\} \quad (13)$$

$$\begin{aligned} \mathcal{P}_{12} = \{ & b_0 a b_1 a^* b_2 a b_3 a^* b_4 \cdots a b_{2k-1} a^* b_{2k} \\ & - \mathcal{E}(b_0 a b_1 a^* b_2 a b_3 a^* b_4 \cdots a b_{2k-1} a^* b_{2k}) \mid k \geq 1, b_0, \dots, b_{2k} \in B\} \end{aligned} \quad (14)$$

$$\begin{aligned} \mathcal{P}_{21} = \{ & b_0 a^* b_1 a b_2 a^* b_3 a b_4 \cdots a^* b_{2k-1} a b_{2k} \\ & - \mathcal{E}(b_0 a^* b_1 a b_2 a^* b_3 a b_4 \cdots a^* b_{2k-1} a b_{2k}) \mid k \geq 1, b_0, \dots, b_{2k} \in B\}. \end{aligned} \quad (15)$$

Note that \mathcal{P}_{1x} means “starting with a ” and \mathcal{P}_{2x} means “starting with a^* ”, while \mathcal{P}_{x1} means “ending with a ” and \mathcal{P}_{x2} means “ending with a^* ”.

When we write that a unitary u *normalizes* B in item (c) below, we mean that $ubu^* \in B$ and $u^*bu \in B$ for all $b \in B$.

The next result is essentially, a B -valued version of Theorem 1.2 of [7]. The equivalence of (b), (d) and (g) was proved in [12]; here we will prove the equivalence of (a)–(f).

Theorem 3.1. *Let $a \in A$. Then the following are equivalent:*

(a) *a is B -valued R -diagonal.*

(b) *We have*

$$\mathcal{E}(x_1 x_2 \cdots x_n) = 0 \quad (16)$$

whenever $n \in \mathbf{N}$, $i_0, i_1, \dots, i_n \in \{1, 2\}$ and $x_j \in \mathcal{P}_{i_{j-1}, i_j}$.

(c) *There is a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$, an element $p \in \tilde{A}$ and a unitary $u \in \tilde{A}$ such that*

(i) *u normalizes B ,*

(ii) *$\tilde{\mathcal{E}}(p) = 0$ and, for all $k \geq 1$ and all $b_1, \dots, b_{2k} \in B$, we have*

$$\tilde{\mathcal{E}}(p b_1 p^* b_2 p b_3 p^* b_4 \cdots p b_{2k-1} p^* b_{2k} p) = 0,$$

(iii) *$\{u, u^*\}$ is free from $\{p, p^*\}$ with respect to $\tilde{\mathcal{E}}$,*

(iv) *$\tilde{\mathcal{E}}(u) = 0$,*

(v) *a and up have the same B -valued $*$ -distribution.*

(d) *There is an enlargement $(\tilde{A}, \tilde{\mathcal{E}})$ of (A, \mathcal{E}) and a unitary $u \in \tilde{A}$ such that*

(i) *u commutes with every element of B ,*

(ii) *$\{u, u^*\}$ is free from $\{a, a^*\}$ with respect to $\tilde{\mathcal{E}}$,*

(iii) *$\tilde{\mathcal{E}}(u^k) = 0$ for all $k \in \mathbf{N}$ (namely, u is Haar unitary),*

(iv) *a and ua have the same B -valued $*$ -distribution.*

(e) *If $(\tilde{A}, \tilde{\mathcal{E}})$ is an enlargement of (A, \mathcal{E}) and $u \in \tilde{A}$ is a unitary such that*

(i) *u commutes with every element of B ,*

(ii) *$\{u, u^*\}$ is free from $\{a, a^*\}$ with respect to $\tilde{\mathcal{E}}$,*

then a and ua have the same B -valued $$ -distribution.*

(f) *Consider the subalgebra*

$$B^{(2)} := \left\{ \begin{pmatrix} b^{(1)} & 0 \\ 0 & b^{(2)} \end{pmatrix} \mid b^{(1)}, b^{(2)} \in B \right\} \subseteq M_2(A),$$

the conditional expectation $\mathcal{E}^{(2)} : M_2(A) \rightarrow B^{(2)}$ given by

$$\mathcal{E}^{(2)} \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = \begin{pmatrix} \mathcal{E}(a_{11}) & 0 \\ 0 & \mathcal{E}(a_{22}) \end{pmatrix},$$

the subalgebra $M_2(B) \subseteq M_2(A)$ and the operator

$$z = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}.$$

- Then $\{z\}$ and $M_2(B)$ are free with respect to $\mathcal{E}^{(2)}$ (i.e., with amalgamation over $B^{(2)}$).
 (g) Letting $a_1 = a$ and $a_2 = a^*$, for every $n \in \mathbf{N}$ and $j = (j(1), \dots, j(n)) \in \{1, 2\}^n$, the B -valued cumulant map α_j for the pair (a_1, a_2) is equal to zero if j is either not alternating or is not of even length, i.e., if j is not of the form $(1, 2, 1, 2, \dots, 1, 2)$ or $(2, 1, 2, 1, \dots, 2, 1)$.

The rest of the section is devoted to the proof of (a)–(f).

Proof of (a) \iff (b). This is clear, because every expectation on the left-hand-side of (1) in Definition 1.2 is one of the expectations on the left-hand-side of (16), and vice-versa and because the vanishing of odd, alternating moments of a is equivalent to $\mathcal{P}_{11} \subseteq \ker \mathcal{E}$. \square

Next is a straightforward computation that we will use twice, so we state it as a separate lemma.

Lemma 3.2. *Let (A, \mathcal{E}) be a B -valued $*$ -noncommutative probability space. Assume $u, p \in A$ are such that*

- (i) u a unitary that normalizes B
- (ii) $\{u, u^*\}$ and $\{p, p^*\}$ are free with respect to \mathcal{E} .

Let $a = up$ and let \mathcal{P}_{ij} be as in (12)–(15). Let $\mathfrak{A} = \text{alg}(B \cup \{p, p^*\})$ and $\mathfrak{A}^\circ = \mathfrak{A} \cap \ker \mathcal{E}$. Then

$$\mathcal{P}_{12} \subseteq u\mathfrak{A}^\circ u^*, \quad (17)$$

$$\mathcal{P}_{21} \subseteq \mathfrak{A}^\circ, \quad (18)$$

$$\mathcal{P}_{11} \subseteq u\mathfrak{A}, \quad (19)$$

$$\mathcal{P}_{22} \subseteq \mathfrak{A}u^*. \quad (20)$$

Proof. For $b \in B$, we use the notational convention $b' = u^*bu \in B$. We examine \mathcal{P}_{12} . Consider, for $b_0, \dots, b_{2k} \in B$,

$$y := b_0 a b_1 a^* b_2 a b_3 a^* \cdots b_{2k-2} a b_{2k-1} a^* b_{2k} = b_0 u p b_1 p^* b'_2 p b_3 p^* \cdots b'_{2k-2} p b_{2k-1} p^* b'_{2k} u^*.$$

Letting $\tilde{y} = p b_1 p^* b'_2 p b_3 p^* \cdots b'_{2k-2} p b_{2k-1} p^* b'_{2k}$, using freeness and condition (i), we have

$$\mathcal{E}(y) = b_0 \mathcal{E}(u \tilde{y} u^*) = b_0 u \mathcal{E}(\tilde{y}) u^*.$$

so an arbitrary element of \mathcal{P}_{12} can be written in the form

$$x = y - \mathcal{E}(y) = b_0 u (\tilde{y} - \mathcal{E}(\tilde{y})) u^* = u (b'_0 (\tilde{y} - \mathcal{E}(\tilde{y}))) u^*.$$

We have shown (17).

Proving (18) is even easier. Indeed, we have

$$z := b_0 a^* b_1 a b_2 a^* b_3 a \cdots b_{2k-2} a^* b_{2k-1} a b_{2k} = b_0 p^* b'_1 p b_2 p^* b'_3 p \cdots b_{2k-2} p^* b'_{2k-1} p b_{2k} \in \mathfrak{A},$$

so an arbitrary element of \mathcal{P}_{21} can be written $x = z - \mathcal{E}(z) \in \mathfrak{A}^\circ$. Similarly, an arbitrary element of \mathcal{P}_{11} is of the form

$$b_0 a b_1 a^* b_2 a b_3 \cdots a^* b_{2k} a b_{2k+1} = u b'_0 p b_1 p^* b'_2 p b_3 \cdots p^* b'_{2k} p b_{2k+1} \in u\mathfrak{A}, \quad (21)$$

proving (19). Taking conjugates proves (20). \square

The next lemma is an analogue of Proposition 2.3 of [7].

Lemma 3.3. *Let (A, \mathcal{E}) be a B -valued $*$ -noncommutative probability space. Assume $u, p \in A$ satisfy*

- (i) u is a unitary that normalizes B ,
- (ii) $\{u, u^*\}$ is free from $\{p, p^*\}$ with respect to \mathcal{E} ,
- (iii) $\mathcal{E}(u) = 0$,
- (iv) $\mathcal{E}(p) = 0$ and, for all $k \geq 1$ and all $b_1, \dots, b_{2k} \in B$, we have

$$\mathcal{E}(pb_1p^*b_2pb_3p^*b_4 \cdots pb_{2k-1}p^*b_{2k}p) = 0.$$

Then the element $a = up$ satisfies the condition (b) of Theorem 3.1.

Proof. From Lemma 3.2 we have (17) and (18). Examining (21) in the proof of Lemma 3.2 and invoking condition (iv), (then also taking conjugates), we find

$$\mathcal{P}_{11} \subseteq u\mathfrak{A}^0, \quad (22)$$

$$\mathcal{P}_{22} \subseteq \mathfrak{A}^0 u^*. \quad (23)$$

Using (17)–(18) and (22)–(23), we see that any product of the form $x_1 x_2 \cdots x_n$ with $x_j \in \mathcal{P}_{i_j, i_{j+1}}$ for some $i_1, \dots, i_{n+1} \in \{1, 2\}$ can be rewritten as a word with letters belonging to the sets \mathfrak{A}^0 and $\{u, u^*\}$, in alternating fashion. By freeness and the hypothesis $\mathcal{E}(u) = 0$, each such word evaluates to 0 under \mathcal{E} . \square

Next is an analogue of Lemma 2.5 of [7].

Lemma 3.4. *Suppose $u \in A$ is a B -normalizing Haar unitary element in a B -valued $*$ -noncommutative probability space (A, \mathcal{E}) . Suppose $D \subseteq A$ is a $*$ -subalgebra that is free from $\{u, u^*\}$ with respect to \mathcal{E} . Let $n \geq 1$, $x_1, \dots, x_n \in D$ and $h_0, h_1, \dots, h_n \in \mathbf{Z}$ be such that*

- (i) $h_k \neq 0$ if $1 \leq k \leq n-1$
- (ii) $h_{k-1}h_k \geq 0$ and at least one of h_{k-1} and h_k is nonzero whenever $1 \leq k \leq n$ and $\mathcal{E}(x_k) \neq 0$.

Then

$$\mathcal{E}(u^{h_0}x_1u^{h_1}x_2 \cdots u^{h_{n-1}}x_nu^{h_n}) = 0. \quad (24)$$

Proof. The proof is very similar to the proof found in [7], only slightly different to take B into account. We use induction on the cardinality m , of $\{k \mid \mathcal{E}(x_k) \neq 0\}$. If $m = 0$, then by freeness, (24) holds. If $m > 0$, then letting k be least such that $\mathcal{E}(x_k) \neq 0$, we write

$$\begin{aligned} & \mathcal{E}(u^{h_0}x_1u^{h_1}x_2 \cdots u^{h_{n-1}}x_nu^{h_n}) \\ &= \mathcal{E}(u^{h_0}x_1u^{h_1}x_2 \cdots x_{k-1}u^{h_{k-1}}\mathcal{E}(x_k)u^{h_k}x_{k+1}u^{h_{k+1}} \cdots x_nu^{h_n}) \\ & \quad + \mathcal{E}(u^{h_0}x_1u^{h_1}x_2 \cdots x_{k-1}u^{h_{k-1}}(x_k - \mathcal{E}(x_k))u^{h_k}x_{k+1}u^{h_{k+1}} \cdots x_nu^{h_n}). \end{aligned}$$

By the induction hypothesis, the second term on the right-hand-side equals 0. Letting

$$b = u^{h_{k-1}}\mathcal{E}(x_k)u^{-h_{k-1}},$$

we have $b \in B$ and the first term equals

$$\mathcal{E}(u^{h_0}x_1u^{h_1} \cdots x_{k-2}u^{h_{k-2}}x_{k-1}bu^{h_{k-1}+h_k}x_{k+1}u^{h_{k+1}} \cdots x_nu^{h_n}). \quad (25)$$

We will show that, by induction hypothesis, the above quantity equals 0. Indeed, we have $h_{k-1}h_k \geq 0$ and at most one of h_{k-1} and h_k can be zero, so $h_{k-1} + h_k \neq 0$. If $k < n$ and $\mathcal{E}(x_{k+1}) \neq 0$, then $h_k h_{k+1} \geq 0$ and we conclude $(h_{k-1} + h_k)h_{k+1} \geq 0$. If $k > 1$ and $\mathcal{E}(x_{k-1}b) \neq 0$, then $\mathcal{E}(x_{k-1})b \neq 0$ so $\mathcal{E}(x_{k-1}) \neq 0$. Consequently, $h_{k-2}h_{k-1} \geq 0$. Thus, $h_{k-2}(h_{k-1} + h_k) \geq 0$. If $k > 1$ then we see that the word to which \mathcal{E} is applied in (25)

satisfies the requirements (i)-(ii) and, applying the induction hypothesis, we conclude that this moment is 0. If $k = 1$, then the moment (25) becomes

$$b\mathcal{E}(u^{h_0+h_1}x_2u^{h_2}\cdots x_nu^{h_n})$$

and again, by the induction hypothesis, this is zero. \square

The next result is an analogue of Proposition 2.4 of [7].

Lemma 3.5. *Suppose that (A, \mathcal{E}) is a B -valued $*$ -noncommutative probability space and $u, p \in A$ are such that*

- (i) *u is a B -normalizing Haar unitary element,*
- (ii) *$\{u, u^*\}$ is free from $\{p, p^*\}$ with respect to \mathcal{E} .*

Let $a = up$. Then a satisfies condition (b) of Theorem 3.1.

Proof. Lemma 3.2 applies and we may use the inclusions (17)–(20), where \mathfrak{A} is the algebra generated by $B \cup \{p, p^*\}$. Thus, for arbitrary $n \in \mathbf{N}$, $i_1, \dots, i_{n+1} \in \{1, 2\}$ and $x_j \in \mathcal{P}_{i_j i_{j+1}}$, the product $x_1 x_2 \cdots x_n$ can be re-written in the form $c_0 u^{\epsilon_1} c_1 u^{\epsilon_2} \cdots c_{r-1} u^{\epsilon_r} c_r$ for some $r \geq 0$, some $c_0, \dots, c_r \in \mathfrak{A}$ and some $\epsilon_1, \dots, \epsilon_r \in \{-1, 1\}$, where whenever $\mathcal{E}(c_j) \neq 0$ for some $1 \leq j \leq r-1$, we have $\epsilon_j = \epsilon_{j+1}$ and in the case $r = 0$, we have $\mathcal{E}(c_0) = 0$. Now Lemma 3.4 applies and we conclude $\mathcal{E}(x_1 x_2 \cdots x_n) = 0$. \square

Proof of (b) \iff (c) \iff (d) \iff (e) of Theorem 3.1. We first show (e) \implies (d). There is a B -valued $*$ -noncommutative probability space (\tilde{B}, \mathcal{F}) with a unitary $v \in \tilde{B}$ such that v commutes with every element of B and $\mathcal{F}(v^n) = 0$ for all $n \in \mathbf{N}$. For example, we could take \tilde{B} to be the algebra $B \otimes \mathbf{C}[C_\infty]$, where $\mathbf{C}[C_\infty]$ is the group $*$ -algebra of the cyclic group of infinite order and where $\mathcal{F}(b \otimes x) = b x_e$, where, for $x \in \mathbf{C}[\mathbf{Z}]$, x_e equals the coefficient in x of the identity element $e \in C_\infty$; let c be a generator of C_∞ and denote also by $c \in \mathbf{C}[C_\infty]$ the corresponding element of the group algebra; then $v = 1 \otimes c$ is a unitary with the desired properties. Then we let

$$(\tilde{A}, \tilde{\mathcal{E}}) = (A, \mathcal{E}) *_B (\tilde{B}, \mathcal{F}) \quad (26)$$

be the algebraic (amalgamated) free product of B -valued $*$ -noncommutative probability spaces and let $u \in \tilde{A}$ be the copy of $v \in \tilde{B}$ arising from the free product construction. Then u commutes with every element of B , $\{u, u^*\}$ and $\{a, a^*\}$ are free and $\tilde{\mathcal{E}}(u) = 0$. By (e), the elements a and ua have the same B -valued $*$ -distribution. Therefore, the requirements of (d) are fulfilled.

We now show (d) \implies (c), taking $p = a$. We need only show that condition (ii) of (c) holds, namely, that $\mathcal{E}(a) = 0$ and

$$\mathcal{E}(ab_1 a^* b_2 a b_3 a^* b_4 \cdots ab_{2k-1} a^* b_{2k} a) = 0.$$

Since, by hypothesis, u commutes with every element B and a has the same B -valued $*$ -distribution as ua , we have

$$\mathcal{E}(a) = \tilde{\mathcal{E}}(ua) = \tilde{\mathcal{E}}(u)\mathcal{E}(a) = 0,$$

where the second equality is due to freeness namely, condition (ii) of (d), and the last equality is because $\tilde{\mathcal{E}}(u) = 0$, namely, condition (iii) of (d). Similarly, we have

$$\begin{aligned} \mathcal{E}(ab_1 a^* b_2 a b_3 a^* b_4 \cdots ab_{2k-1} a^* b_{2k} a) &= \tilde{\mathcal{E}}(uab_1 a^* b_2 ab_3 a^* b_4 \cdots ab_{2k-1} a^* b_{2k} a) \\ &= \tilde{\mathcal{E}}(u)\mathcal{E}(ab_1 a^* b_2 ab_3 a^* b_4 \cdots ab_{2k-1} a^* b_{2k} a) = 0, \end{aligned}$$

with the penultimate equality due to freeness. Thus, (c) holds.

The implication (c) \implies (b) follows immediately by appeal to Lemma 3.3.

We now show (b) \implies (e). We suppose $(\tilde{A}, \tilde{\mathcal{E}})$ is an enlargement of (A, \mathcal{E}) and $u \in A$ is a unitary satisfying (i) and (ii) of (e). We must show that a and ua have the same B -valued $*$ -distribution. Taking $(\tilde{B}, \tilde{\mathcal{F}})$ as in the proof of (e) \implies (d) above and replacing $(\tilde{A}, \tilde{\mathcal{E}})$ by the algebraic free product with amalgamation $(\tilde{A}, \tilde{\mathcal{E}}) *_B (\tilde{B}, \tilde{\mathcal{F}})$, we may assume there exists a Haar unitary $v \in \tilde{A}$ such that v commutes with every element of B and $\{v, v^*\}$ is free from $\{a, u, a^*, u^*\}$. Thus, the triple

$$\{v, v^*\}, \{u, u^*\}, \{a, a^*\}$$

forms a free family of sets.

We make the following claims, which we will prove one after the other:

- (A) a and va have the same B -valued $*$ -distribution,
- (B) ua and uva have the same B -valued $*$ -distribution,
- (C) each of uva and ua satisfies condition (b) of Theorem 3.1,
- (D) a and ua have the same B -valued $*$ -distribution.

To prove (A), we note that both a and va satisfy condition (b) of Theorem 3.1; the operator a does so by hypothesis, and the operator va does so by Lemma 3.5. Moreover, for each $k \geq 1$ and $b_1, \dots, b_{2k-1} \in B$, we have

$$\begin{aligned} \tilde{\mathcal{E}}((va)b_1(va)^*b_2(va)b_3(va)^*b_4 \cdots (va)b_{2k-1}(va)^*) \\ = \tilde{\mathcal{E}}(vab_1a^*b_2ab_3a^*b_4 \cdots ab_{2k-1}a^*v^*) = \tilde{\mathcal{E}}(ab_1a^*b_2ab_3a^*b_4 \cdots ab_{2k-1}a^*), \end{aligned} \quad (27)$$

where the last inequality is due to freeness, and

$$\tilde{\mathcal{E}}((va)^*b_1(va)b_2(va)^*b_3(va)b_4 \cdots (va)^*b_{2k-1}(va)) = \tilde{\mathcal{E}}(a^*b_1ab_2a^*b_3ab_4 \cdots a^*b_{2k-1}a). \quad (28)$$

Thus, by Remark 1.4, a and va have the same B -valued $*$ -distribution, and the claim is proved.

To prove (B), note that $\{u, u^*\}$ is free from $\{a, a^*\}$ and from $\{va, (va)^*\}$ and, by (A), a and va are identically $*$ -distributed. This implies (B).

To prove (C), we argue from Lemma 3.5 that uva satisfies condition (b) of Theorem 3.1, because uv is a B -normalizing Haar unitary. That ua satisfies condition (b) of Theorem 3.1 now follows from (B).

To prove (D), note that the analogues of (27) and (28) hold when v is replaced by u , by the same arguments as given above. Also, both a and ua satisfy condition (b) of Theorem 3.1; the operator a does so by hypothesis, and the operator ua does so by (C). By Remark 1.4, a and ua are identically distributed. This finishes the proof of (b) \implies (e).

This completes the proof of the equivalence of (b), (c), (d) and (e) of Theorem 3.1. \square

For the next lemma, we use the sets \mathcal{P}_{11} and \mathcal{P}_{22} as in (12) and (13), and we use additionally the notation

$$\tilde{\mathcal{P}}_{12} = B \cup \{b_0ab_1a^*b_2ab_3a^*b_4 \cdots ab_{2k-1}a^*b_{2k} \mid k \geq 1, b_0, \dots, b_{2k} \in B\}$$

$$\tilde{\mathcal{P}}_{21} = B \cup \{b_0a^*b_1ab_2a^*b_3ab_4 \cdots a^*b_{2k-1}ab_{2k} \mid k \geq 1, b_0, \dots, b_{2k} \in B\},$$

so that we have

$$\mathcal{P}_{12} = \{w - \mathcal{E}(w) \mid w \in \tilde{\mathcal{P}}_{12}\}, \quad \mathcal{P}_{21} = \{w - \mathcal{E}(w) \mid w \in \tilde{\mathcal{P}}_{21}\}. \quad (29)$$

Now we turn to condition (f) of Theorem 3.1 and take z and $B^{(2)}$ as defined there.

Lemma 3.6. *Let $R \subseteq M_2(A)$ be the subalgebra generated by $\{z\} \cup B^{(2)}$. Then*

$$R = \text{span} \left\{ \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \mid r_{11} \in \tilde{\mathcal{P}}_{12}, r_{12} \in \mathcal{P}_{11}, r_{21} \in \mathcal{P}_{22}, r_{22} \in \tilde{\mathcal{P}}_{21} \right\}. \quad (30)$$

Proof. For $b_1^{(j)}, b_2^{(j)} \in B$, we have

$$\begin{aligned} z \begin{pmatrix} b_1^{(1)} & 0 \\ 0 & b_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 & ab_2^{(1)} \\ a^*b_1^{(1)} & 0 \end{pmatrix} \\ z \begin{pmatrix} b_1^{(1)} & 0 \\ 0 & b_2^{(1)} \end{pmatrix} z \begin{pmatrix} b_1^{(2)} & 0 \\ 0 & b_2^{(2)} \end{pmatrix} &= \begin{pmatrix} ab_2^{(1)}a^*b_1^{(2)} & 0 \\ 0 & a^*b_1^{(1)}ab_2^{(2)} \end{pmatrix}. \end{aligned}$$

Using induction on n , we easily see that every word of the form

$$\begin{pmatrix} b_1^{(0)} & 0 \\ 0 & b_2^{(0)} \end{pmatrix} z \begin{pmatrix} b_1^{(1)} & 0 \\ 0 & b_2^{(1)} \end{pmatrix} z \begin{pmatrix} b_1^{(2)} & 0 \\ 0 & b_2^{(2)} \end{pmatrix} \cdots z \begin{pmatrix} b_1^{(n)} & 0 \\ 0 & b_2^{(n)} \end{pmatrix} \quad (31)$$

belongs to the right-hand-side of (30), from which we easily deduce that the inclusion \subseteq in (30) holds. The reverse inclusion is easily proved by judicious choice of $b_1^{(j)}$ and $b_2^{(j)}$ in (31). \square

Proof of (b) \iff (f) of Theorem 3.1. From Lemma 3.6 and (29), we have

$$R \cap \ker \mathcal{E}^{(2)} = \text{span} \left\{ \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \mid r_{11} \in \mathcal{P}_{12}, r_{12} \in \mathcal{P}_{11}, r_{21} \in \mathcal{P}_{22}, r_{22} \in \mathcal{P}_{21} \right\}. \quad (32)$$

and, clearly, we have

$$M_2(B) \cap \ker \mathcal{E}^{(2)} = \left\{ \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} \mid b_1, b_2 \in B \right\}.$$

To show (b) \implies (f), take $r^{(1)}, \dots, r^{(n)} \in R \cap \ker \mathcal{E}^{(2)}$ and

$$b^{(j)} = \begin{pmatrix} 0 & b_1^{(j)} \\ b_2^{(j)} & 0 \end{pmatrix} \in M_2(B) \cap \ker \mathcal{E}^{(2)}$$

for $0 \leq j \leq n$ and consider the product $r^{(1)}b^{(1)} \dots r^{(n)}b^{(n)}$. Since each $r^{(j)}b^{(j)}$ is a 2×2 matrix whose (k, l) -th entry belonging to P_{kl} , for each $k, l \in \{1, 2\}$, we see that every entry of the 2×2 matrix $r^{(1)}b^{(1)} \dots r^{(n)}b^{(n)}$ is a sum of products of the form of the form that condition (b) of Theorem 3.1 guarantees has expectation zero. Thus, every entry of the 2×2 matrix $r^{(1)}b^{(1)} \dots r^{(n)}b^{(n)}$ evaluates to zero under \mathcal{E} . The same remains true if we left multiply by $b^{(0)}$ or if we choose $b_1^{(n)} = b_2^{(n)} = 1$ and then right multiply by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or if we do both. This shows that each of the products

$$\begin{aligned} r^{(1)}b^{(1)} \dots r^{(n)}b^{(n)}, \quad r^{(1)}b^{(1)}r^{(2)} \dots b^{(n-1)}r^{(n)}, \\ b^{(0)}r^{(1)}b^{(1)} \dots r^{(n)}b^{(n)}, \quad b^{(0)}r^{(1)} \dots b^{(n-1)}r^{(n)}, \end{aligned} \quad (33)$$

expects to zero under $\mathcal{E}^{(2)}$. This is (f) of Theorem 3.1.

To show (f) \implies (b), using (32), it is straightforward to arrange, given any $i_1, \dots, i_{n+1} \in \{1, 2\}$ and any $x_j \in \mathcal{P}_{i_j i_{j+1}}$, that the product $x_1 x_2 \dots x_n$ arise as either the $(1, 1)$ or $(2, 2)$ entry of a product of one of the forms (33) for suitable $r^{(1)}, \dots, r^{(n)}$ and for each $b^{(j)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus, $\mathcal{E}(x_1 x_2 \dots x_n) = 0$. \square

Since the equivalence (g) \iff (b) from Theorem 3.1 was proved in [12], the proof of Theorem 3.1 is complete.

4. FORMAL POWER SERIES

In this section, we derive some formal power series relations involving moments and cumulants of R-diagonal elements. The main result is essentially a variation on the combinatorial proof of the power series relation for the R-transform, here modified to handle R-diagonal elements. The result is of intrinsic interest and will be used in the appendix to investigate the particular operator considered in Example 6.9.

Throughout this section, B will be a unital $*$ -algebra, (A, \mathcal{E}) will be a B -valued $*$ -noncommutative probability space and $a \in A$ will be a B -valued R-diagonal element. We let $a_1 = a$ and $a_2 = a^*$ and let α_j for sequences j in $\{1, 2\}$ denote the B -valued cumulant maps of the pair (a_1, a_2) . For $k \in \mathbb{N}$, for the $2k - 1$ -multilinear cumulant maps that are the only ones that need not be zero, we write

$$\alpha_k^{(1)} = \alpha_{\underbrace{(1, 2, 1, 2, \dots, 1, 2)}_{2k}}, \quad \alpha_k^{(2)} = \alpha_{\underbrace{(2, 1, 2, 1, \dots, 2, 1)}_{2k}}.$$

Proposition 4.1. *For $k \in \mathbb{N}$ and $b_1, \dots, b_{2k} \in B$, let us write*

$$\begin{aligned} m_k^{(1)}(b_1, \dots, b_{2k}) &= \mathcal{E}(ab_1a^*b_2ab_3a^*b_4 \cdots ab_{2k-1}a^*b_{2k}) \\ m_k^{(2)}(b_1, \dots, b_{2k}) &= \mathcal{E}(a^*b_1ab_2a^*b_3ab_4 \cdots a^*b_{2k-1}ab_{2k}), \end{aligned}$$

and

$$m_0^{(1)} = m_0^{(2)} = 1 \in B.$$

Then, for every $n \geq 1$ and $b_1, \dots, b_{2n} \in B$, we have

$$\begin{aligned} m_n^{(1)}(b_1, \dots, b_{2n}) &= \sum_{\ell=1}^n \sum_{\substack{k(1), \dots, k(2\ell) \geq 0 \\ k(1) + \dots + k(2\ell) = n - \ell}} \alpha_\ell^{(1)} \left(b_{s(1)+1} m_{k(1)}^{(2)}(b_{s(1)+2}, \dots, b_{s(1)+2k(1)+1}), \right. \\ &\quad b_{s(2)+2} m_{k(2)}^{(1)}(b_{s(2)+3}, \dots, b_{s(2)+2k(2)+2}), \\ &\quad b_{s(3)+3} m_{k(3)}^{(2)}(b_{s(3)+4}, \dots, b_{s(3)+2k(3)+3}), \\ &\quad \vdots \\ &\quad b_{s(2\ell-2)+2\ell-2} m_{k(2\ell-2)}^{(1)}(b_{s(2\ell-2)+2\ell-1}, \dots, b_{s(2\ell-2)+2k(2\ell-2)+2\ell-2}), \\ &\quad b_{s(2\ell-1)+2\ell-1} m_{k(2\ell-1)}^{(2)}(b_{s(2\ell-1)+2\ell}, \dots, b_{s(2\ell-1)+2k(2\ell-1)+2\ell-1}) \\ &\quad \left. \right) b_{s(2\ell)+2\ell} m_{k(2\ell)}^{(1)}(b_{s(2\ell)+2\ell+1}, \dots, b_{s(2\ell)+2k(2\ell)+2\ell}) \end{aligned} \tag{34}$$

and

$$\begin{aligned}
m_n^{(2)}(b_1, \dots, b_{2n}) &= \sum_{\ell=1}^n \sum_{\substack{k(1), \dots, k(2\ell) \geq 0 \\ k(1) + \dots + k(2\ell) = n - \ell}} \\
&\alpha_\ell^{(2)} \left(b_{s(1)+1} m_{k(1)}^{(1)}(b_{s(1)+2}, \dots, b_{s(1)+2k(1)+1}), \right. \\
&\quad b_{s(2)+2} m_{k(2)}^{(2)}(b_{s(2)+3}, \dots, b_{s(2)+2k(2)+2}), \\
&\quad b_{s(3)+3} m_{k(3)}^{(1)}(b_{s(3)+4}, \dots, b_{s(3)+2k(3)+3}), \\
&\quad \vdots \\
&\quad b_{s(2\ell-2)+2\ell-2} m_{k(2\ell-2)}^{(2)}(b_{s(2\ell-2)+2\ell-1}, \dots, b_{s(2\ell-2)+2k(2\ell-2)+2\ell-2}), \\
&\quad b_{s(2\ell-1)+2\ell-1} m_{k(2\ell-1)}^{(1)}(b_{s(2\ell-1)+2\ell}, \dots, b_{s(2\ell-1)+2k(2\ell-1)+2\ell-1}) \\
&\quad \left. \right) b_{s(2\ell)+2\ell} m_{k(2\ell)}^{(2)}(b_{s(2\ell)+2\ell+1}, \dots, b_{s(2\ell)+2k(2\ell)+2\ell}),
\end{aligned}$$

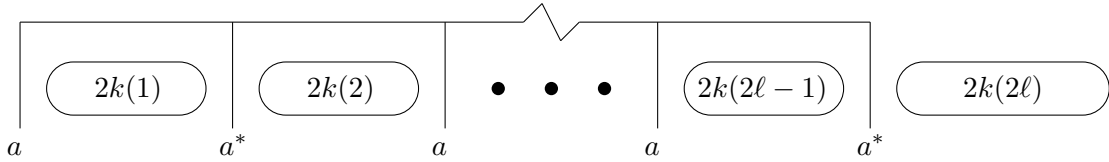
where $s(j) = 2(k(1) + \dots + k(j-1))$, which includes the case $s(1) = 0$.

Proof. Using the moment-cumulant formula, we have

$$m_n^{(1)}(b_1, \dots, b_{2n}) = \sum_{\pi \in \text{NC}(2n)} \hat{\alpha}_{(1,2,\dots,1,2)}(\pi) [b_1, \dots, b_{2n-1}] b_{2n}.$$

If $\pi \in \text{NC}(2n)$ yields a nonzero term in the above sum, then the block of π containing 1 must contain alternating odd and even numbers increasing from left to right. Such a block is indicated in in Figure 1. The ovals represent the locations of the other blocks of the

FIGURE 1. A noncrossing partition for an alternating $*$ -moment.



partition, and the quantities $2k(j)$ in the ovals are the lengths of the respective gaps, which may be zero. If we sum the quantity

$$\hat{\alpha}_{(1,2,\dots,1,2)}(\pi) [b_1, \dots, b_{2n-1}] b_{2n} \quad (35)$$

over all partitions $\pi \in \text{NC}(2n)$ whose block containing 1 is the given one shown in the figure, then by the moment-cumulant formula applied to each of the 2ℓ ovals, we obtain precisely the summand of the summation appearing in (34). Now fixing ℓ and summing this value over all possible values of $k(1), \dots, k(2\ell)$ equals the sum of the quantity (35) over all $\pi \in \text{NC}(2n)$ whose first block contains 2ℓ elements. Finally, summing over all values of ℓ yields the equality (34).

The other equality is proved in the same way, by changing indices. \square

Theorem 4.2. *Consider the formal power series*

$$F(b_1, b_2) = \sum_{n=0}^{\infty} \mathcal{E}((ab_1a^*b_2)^n) \quad (36)$$

$$G(b_1, b_2) = \sum_{n=0}^{\infty} \mathcal{E}((a^*b_1ab_2)^n). \quad (37)$$

Then

$$F(b_1, b_2) = 1 + \sum_{\ell=1}^{\infty} \alpha_{\ell}^{(1)} \left(b_1 G(b_2, b_1), b_2 F(b_1, b_2), b_1 G(b_2, b_1), \right. \\ \left. \dots, b_2 F(b_1, b_2), b_1 G(b_2, b_1) \right) b_2 F(b_1, b_2), \quad (38)$$

$$G(b_1, b_2) = 1 + \sum_{\ell=1}^{\infty} \alpha_{\ell}^{(2)} \left(b_1 F(b_2, b_1), b_2 G(b_1, b_2), b_1 F(b_2, b_1), \right. \\ \left. \dots, b_2 G(b_1, b_2), b_1 F(b_2, b_1) \right) b_2 G(b_1, b_2). \quad (39)$$

The meaning of the above formulas should be clear. In (36) and (37), the series are formal power series in variables b_1 and b_2 , and for a given n , the corresponding terms should be thought of as being of degree $2n$. In the formulas (38) and (39), the formulas on the right-hand-sides mean the formal power series obtained by substituting and formally expanding. Since the degree $2n$ term in (36) is precisely $m^{(1)}(b_1, b_2, b_1, b_2, \dots, b_1, b_2)$, and in (37) it is similarly $m^{(2)}(b_1, b_2, b_1, b_2, \dots, b_1, b_2)$, the assertions (38)–(39) follow from Proposition 4.1.

The expressions in (38)–(39) can be treated more precisely and in a more general framework using the notion of multilinear function series that was described in [2]. Formally, a *multilinear function series* over B is a family $X = (\chi_n)_{n \geq 0}$ where $\chi_0 \in B$ is the constant term and where, when $n \geq 1$, χ_n is a n -fold multilinear function $\chi_n : B^n \rightarrow B$. The set $\text{Mul}[[B]]$ of all multilinear function series over B was shown in [2] to be an associative ring and a B -bimodule under natural operations. For example, the product is given by $X\Psi$, for X as above and $\Psi = (\psi_n)_{n \geq 0} \in \text{Mul}[[B]]$, where the 0-th term of $X\Psi$ is $\chi_0\psi_0$ and the n -th term, for $n \geq 1$, is the multilinear map

$$(b_1, \dots, b_n) \mapsto \sum_{k=0}^n \chi_k(b_1, \dots, b_k) \psi_{n-k}(b_{k+1}, \dots, b_n).$$

Moreover, $\text{Mul}[[B]]$ is equipped with a composition operation, (which will be generalized below). The identity element of $\text{Mul}[[B]]$ with respect to composition is denoted I ; it is the multilinear function series whose n -th term is 0 unless $n = 1$, in which case it is the identity map on B .

Definition 4.3. Let $(b_i)_{i=1}^{\infty}$ be a sequence in B . Let us say that a *variable assignment* for a multilinear function series $X = (\chi_n)_{n=0}^{\infty}$ is a choice $v : \{(n, j) \in \mathbf{N}^2 \mid n \geq j\} \rightarrow B$. Then the *evaluation* of X at v is just the formal sum

$$X(v) = \chi_0 + \sum_{n=1}^{\infty} \chi_n(v(n, 1), \dots, v(n, n)).$$

Note that, though this is ostensibly a summation of elements of B , no sort of convergence is assumed. Formally, it may be thought of as the sequence

$$(\chi_0, \chi_1(v(1, 1)), \chi_2(v(2, 1), v(2, 2)), \dots)$$

in B , and equality of such formal sums amounts to equality of the corresponding sequences.

If $M^{(i)} \in \text{Mul}[[B]]$ is the multilinear function series whose 0-th term is 1 and whose $2n$ -th term for $n \geq 1$ is $m_n^{(i)}$ as described in Proposition 4.1, and if

$$v(n, j) = \begin{cases} b_1, & j \text{ odd} \\ b_2, & j \text{ even}, \end{cases} \quad (40)$$

then in (36)–(37), F is just $M^{(1)}$ evaluated at v and G is just $M^{(2)}$ evaluated at v .

Definition 4.4. Suppose that $(\Psi^{(i)})_{i \in \mathbf{N}}$ is a family in $\text{Mul}[[B]]$, where each $\Psi^{(i)} = (\psi_n^{(i)})_{n \geq 0}$ has zero constant term and suppose that $X = (\chi_n)_{n \geq 0} \in \text{Mul}[[B]]$. Let $f : \{(n, j) \in \mathbf{N}^2 \mid n \geq j\} \rightarrow \mathbf{N}$ be a function. Then the *multivariate composition*

$$X \overset{f}{\circ} (\Psi^{(i)})_{i \in \mathbf{N}}$$

is obtained by, heuristically, replacing the j -th argument of χ_n by $\Psi^{(f(n, j))}$ for each n and j . More precisely, it is the element of $\text{Mul}[[B]]$ whose n -th term is χ_0 when $n = 0$ and when $n \geq 1$ it is the n -fold multilinear function that sends (b_1, \dots, b_n) to

$$\sum_{p=1}^n \sum_{\substack{k(1), \dots, k(p) \geq 1 \\ k(1) + \dots + k(p) = n}} \chi_p(\psi_{k(1)}^{(f(p, 1))}(b_1, \dots, b_{k(1)}), \psi_{k(2)}^{(f(p, 2))}(b_{k(1)+1}, \dots, b_{k(1)+k(2)}), \dots, \psi_{k(p)}^{(f(p, p))}(b_{k(1)+\dots+k(p-1)+1}, \dots, b_{k(1)+\dots+k(p-1)+k(p)})).$$

The following result is a rephrasing of Proposition 4.1. Note that the product $IM^{(i)}$ is given by $(IM^{(i)})_n = 0$ if n is even and

$$(IM^{(i)})_{2n+1}(b_1, \dots, b_{2n+1}) = \begin{cases} b_1 m_n^{(i)}(b_2, b_3, \dots, b_{2n+1}), & n \geq 1 \\ b_1, & n = 0. \end{cases}$$

Proposition 4.5. For $i = 1, 2$, let $A^{(i)}$ be the multilinear function series with zero constant term, whose even terms vanish and whose $(2\ell - 1)$ -th term is $\alpha_\ell^{(i)}$. Let $f, g : \{(n, j) \in \mathbf{N}^2 \mid n \geq j\} \rightarrow \{1, 2\}$ be given by $f(n, j) \equiv j + 1 \pmod{2}$ and $g(n, j) \equiv j \pmod{2}$. Then

$$M^{(1)} = 1 + (A^{(1)} \overset{f}{\circ} (IM^{(1)}, IM^{(2)})) M^{(1)} \quad (41)$$

$$M^{(2)} = 1 + (A^{(2)} \overset{g}{\circ} (IM^{(1)}, IM^{(2)})) M^{(2)}. \quad (42)$$

Now (38)–(39) result from evaluating both sides of (41)–(42) at v , as given in (40). This, then, is a formal interpretation of Theorem 4.2.

5. TRACES AND POLAR DECOMPOSITION

In this section, we suppose that B is a unital $*$ -algebra and (A, \mathcal{E}) is a B -valued $*$ -noncommutative probability space. We investigate questions of traces and polar decomposition for B -valued R-diagonal elements.

First some notation we will use in this section: if $a \in A$, then we write $a_1 = a$ and $a_2 = a^*$ and for $k \in \mathbf{N}$, let $\beta_k^{(1)}$ and $\beta_k^{(2)}$ be the multilinear maps

$$\beta_k^{(i)} : \underbrace{B \times \dots \times B}_{2k-1} \rightarrow B \quad (43)$$

that are the cumulant maps for the pair (a_1, a_2) corresponding to the alternating sequences $(1, 2, \dots, 1, 2)$ and $(2, 1, \dots, 2, 1)$, respectively, each of length $2k$. Note that, from Proposition 2.4, we have

$$\beta_k^{(i)}(b_1, \dots, b_{2k-1})^* = \beta_k^{(i)}(b_{2k-1}^*, \dots, b_1^*) \quad (44)$$

for all $i = 1, 2$, $k \geq 1$ and $b_1, \dots, b_{2k-1} \in B$.

The following result is an immediate consequence of Proposition 2.2.

Proposition 5.1. *Suppose $a \in A$ is B -valued R -diagonal. Suppose τ is a tracial linear functional on B . Then $\tau \circ \mathcal{E}$ restricted to $\text{alg}(B \cup \{a, a^*\})$ is tracial if and only if, for all $k \geq 1$ and all $b_1, \dots, b_{2k} \in B$, we have*

$$\tau(\beta_k^{(1)}(b_1, \dots, b_{2k-1})b_{2k}) = \tau(b_1\beta_k^{(2)}(b_2, \dots, b_{2k})).$$

We now turn to results about B -valued R -diagonal elements written in the form up where u is unitary and where $p = p^*$. First, we make an observation about the polar decompositions of R -diagonal elements in W^* -noncommutative probability spaces.

Proposition 5.2. *Suppose B is a von Neumann algebra and (A, \mathcal{E}) is a B -valued W^* -noncommutative probability space. Suppose $a \in A$ is B -valued R -diagonal and $a = v|a|$ is the polar decomposition of a . Then $\mathcal{E}(v^n) = 0$ for all integers $n \geq 1$. Thus, if v is unitary, then it is Haar unitary.*

Proof. Using the amalgamated free product construction for von Neumann algebras, we get a larger W^* -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ containing a Haar unitary u that commutes with every element of B and such that $\{a, a^*\}$ and $\{u, u^*\}$ are free with respect to $\tilde{\mathcal{E}}$. Thus, $\{|a|, v, v^*\}$ and $\{u, u^*\}$ are free with respect to $\tilde{\mathcal{E}}$. By condition (e) of Theorem 3.1, ua has the same B -valued $*$ -distribution with respect to $\tilde{\mathcal{E}}$ as does a with respect to \mathcal{E} . The polar decomposition of ua is $uv|a|$. Since we are in a W^* -noncommutative probability space, given any element x with polar decomposition $x = w|x|$, the joint B -valued $*$ -distribution of the pair $(w, |x|)$ is determined by the B -valued $*$ -distribution of x . Therefore, the B -valued $*$ -distribution of uv equals that of v . Now, using that u is a Haar unitary that commutes with B and that u and v are $*$ -free, it is straightforward to show $\tilde{\mathcal{E}}((uv)^n) = 0$ for every integer $n \geq 1$. Thus, also $\tilde{\mathcal{E}}(v^n) = 0$. \square

Proposition 2.6 of [7] shows that in the scalar-valued case, given an R -diagonal element in a tracial $*$ -noncommutative probability space, it has the same $*$ -distribution as some element up , where u is a Haar unitary, p is self-adjoint and u is $*$ -free from p . The following observation will be used in Example 6.9 below, to show that the analogous statement need not hold in the algebra-valued case.

Lemma 5.3. *Suppose in a B -valued $*$ -noncommutative probability space (A, \mathcal{E}) , $a = up \in A$ is B -valued R -diagonal, where $u \in A$ is unitary, where $p = p^* \in A$ and where $\{u, u^*\}$ and $\{p\}$ are free (over B) with respect to \mathcal{E} . If $\beta_2^{(1)}(1) \in \mathbf{C}1$, then $\beta_1^{(1)}(1) = \beta_2^{(1)}(1)$.*

Proof. We have $\beta_1^{(2)}(1) = \mathcal{E}(a^*a) = \mathcal{E}(p^2)$, while using freeness of $\{u, u^*\}$ and $\{p\}$, we find

$$\beta_1^{(1)}(1) = \mathcal{E}(up^2u^*) = \mathcal{E}(u\mathcal{E}(p^2)u^*) = \beta_2^{(1)}(1)\mathcal{E}(uu^*) = \beta_2^{(1)}(1)\mathcal{E}(1) = \beta_2^{(1)}(1).$$

\square

We now study R -diagonal elements that can be written in the form up where u is a B -normalizing Haar unitary, $p = p^*$ and the sets $\{u, u^*\}$ and $\{p\}$ are free over B . It turns out these can be characterized in terms of cumulants. Our study culminates in Theorem 5.8

and Corollary 5.10, which apply to C^* - and W^* -noncommutative probability spaces, but along the way we prove algebraic versions too.

The following proposition includes an analogue of Proposition 2.6 of [7]. Note that the equivalence of parts (a)–(d) of it are special cases of Proposition 5.5, below. However, we include the statment and proof of it, because both are easier than the more general case, and serve as templates.

Proposition 5.4. *Let $a \in A$. Then the following are equivalent:*

(a) *a is B -valued R -diagonal and for all $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$, we have*

$$\mathcal{E}(a^*b_1ab_2a^*b_3a \cdots b_{2k-2}a^*b_{2k-1}a) = \mathcal{E}(ab_1a^*b_2ab_3a^* \cdots b_{2k-2}ab_{2k-1}a^*). \quad (45)$$

(b) *a is B -valued R -diagonal and for all $k \geq 1$, we have agreement $\beta_k^{(1)} = \beta_k^{(2)}$ of the $2k$ -th order B -valued cumulant maps*

(c) *There exists a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ with elements $p, u \in \tilde{A}$ such that*

(i) $p = p^*$

(ii) u is a Haar unitary that commutes with every element of B

(iii) $\{u, u^*\}$ and $\{p\}$ are free (over B) with respect to \mathcal{E}

(iv) the element up has the same B -valued $*$ -distribution as a

(v) the odd moments of p vanish, i.e., for every $k \geq 0$ and all $b_1, \dots, b_{2k} \in B$, we have

$$\tilde{\mathcal{E}}(pb_1pb_2 \cdots pb_{2k}p) = 0. \quad (46)$$

(d) *There exists a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ with elements $p, u \in \tilde{A}$ such that parts (i)–(iv) of (c) hold.*

(e) *a is B -valued R -diagonal and has the same B -valued $*$ -distribution as a^* .*

In part (c) or (d), the even moments of p are given by, for every $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$,

$$\tilde{\mathcal{E}}(pb_1pb_2 \cdots pb_{2k-1}p) = \mathcal{E}(a^*b_1ab_2a^*b_3a \cdots b_{2k-2}a^*b_{2k-1}a). \quad (47)$$

Proof. The equivalence of (e), (a) and (b) is clear from the moment-cumulant formula and Remark 1.4. Indeed, using Remark 1.3, condition (b) says that a and a^* have the same B -valued cumulants, since all other cumulants of B -valued R -diagonal elements must vanish.

We now show (b) \implies (c). Endow $B\langle X \rangle$ with the $*$ -operation that extends the given one on B and satisfies $X^* = X$. Let $\Theta : B\langle X \rangle \rightarrow B$ be the B -valued distribution of an element whose odd order cumulant maps are all zero and whose B -valued cumulant map of order $2k$ is $\beta_k^{(1)}$, for all $k \geq 1$; indeed, Θ is constructed using the moment-cumulant formula. Write p for the element X of $B\langle X \rangle$. By Proposition 2.4 and the identity (44), Θ is a self-adjoint map. By the amalgamated free product construction, as in (26), there is a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ containing elements $p, u \in \tilde{A}$ satisfying conditions (i), (ii) and (iii) of (c). The moment-cumulant formula and the fact that $\beta_k^{(1)} = \beta_k^{(2)}$ for all k imply that the identities (46) and (47) hold; in particular, condition (v) of (c) holds. Let $\tilde{a} = up \in \tilde{A}$. By Lemma 3.3, \tilde{a} is B -valued R -diagonal. In order to prove that a and \tilde{a} have the same B -valued $*$ -distribution, using Remark 1.4, it will suffice to show that we always have

$$\mathcal{E}(a^*b_1ab_2a^*b_3ab_4 \cdots a^*b_{2k-1}a) = \tilde{\mathcal{E}}(\tilde{a}^*b_1\tilde{a}b_2\tilde{a}^*b_3\tilde{a}b_4 \cdots \tilde{a}^*b_{2k-1}\tilde{a}) \quad (48)$$

$$\mathcal{E}(ab_1a^*b_2ab_3a^*b_4 \cdots ab_{2k-1}a^*) = \tilde{\mathcal{E}}(\tilde{a}b_1\tilde{a}^*b_2\tilde{a}b_3\tilde{a}^*b_4 \cdots \tilde{a}b_{2k-1}\tilde{a}^*). \quad (49)$$

To show (48), we use

$$\tilde{a}^* b_1 \tilde{a} b_2 \tilde{a}^* b_3 \tilde{a} b_4 \cdots \tilde{a}^* b_{2k-1} \tilde{a} = p b_1 p b_2 \cdots p b_{2k-1} p, \quad (50)$$

and (47). To show (49), we use

$$\tilde{a} b_1 \tilde{a}^* b_2 \tilde{a} b_3 \tilde{a}^* b_4 \cdots \tilde{a} b_{2k-1} \tilde{a}^* = u(p b_1 p b_2 \cdots p b_{2k-1} p) u^*.$$

By freeness and the fact that u commutes with every element of B , we get

$$\tilde{\mathcal{E}}(\tilde{a} b_1 \tilde{a}^* b_2 \tilde{a} b_3 \tilde{a}^* b_4 \cdots \tilde{a} b_{2k-1} \tilde{a}^*) = \tilde{\mathcal{E}}(p b_1 p b_2 \cdots p b_{2k-1} p) \quad (51)$$

so, by appeal to (47) and the fact that a and a^* have the same B -valued $*$ -distribution, we get (49). Thus, a and $\tilde{a} = up$ have the same B -valued $*$ -distribution; namely, (c) holds.

Clearly, (c) \implies (d).

Now we will show (d) implies (a) and (47). Let $\tilde{a} = up$. That \tilde{a} is B -valued R-diagonal follows from Lemma 3.5. But (50) and (51) hold by the same arguments as above. Thus, we have (45). \square

Here is a more general version of Proposition 5.4.

Proposition 5.5. *Let $a \in A$ and let θ be an automorphism of the $*$ -algebra B . Then the following are equivalent:*

(a) *a is B -valued R-diagonal and for all $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$, we have*

$$\begin{aligned} \mathcal{E}(a^* b_1 a \theta(b_2) a^* b_3 a \cdots \theta(b_{2k-2}) a^* b_{2k-1} a) \\ = \theta(\mathcal{E}(a \theta(b_1) a^* b_2 a \theta(b_3) a^* \cdots b_{2k-2} a \theta(b_{2k-1}) a^*)). \end{aligned} \quad (52)$$

(b) *a is B -valued R-diagonal and for all $k \geq 1$, the $2k$ -th order B -valued cumulant maps satisfy, for all $b_1, \dots, b_{2k-1} \in B$,*

$$\beta_k^{(2)}(b_1, \theta(b_2), b_3, \dots, \theta(b_{2k-2}), b_{2k-1}) = \theta(\beta_k^{(1)}(\theta(b_1), b_2, \theta(b_3), \dots, b_{2k-2}, \theta(b_{2k-1}))). \quad (53)$$

(c) *There exists a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ with elements $p, u \in \tilde{A}$ such that*

- (i) $p = p^*$
- (ii) u is a Haar unitary that normalizes B and satisfies $u^* b u = \theta(b)$ for all $b \in B$
- (iii) $\{u, u^*\}$ and $\{p\}$ are free (over B) with respect to \mathcal{E}
- (iv) the element up has the same B -valued $*$ -distribution as a
- (v) the odd moments of p vanish, i.e., for every $k \geq 0$ and all $b_1, \dots, b_{2k} \in B$, we have

$$\tilde{\mathcal{E}}(p b_1 p b_2 \cdots p b_{2k} p) = 0.$$

(d) *There exists a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ with elements $p, u \in \tilde{A}$ such that parts (i)–(iv) of (c) hold.*

In part (c) or (d), the even moments of p are given by, for every $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$,

$$\tilde{\mathcal{E}}(p b_1 p b_2 \cdots p b_{2k-1} p) = \mathcal{E}(a^* \theta^{-1}(b_1) a b_2 a^* \theta^{-1}(b_3) a \cdots b_{2k-2} a^* \theta^{-1}(b_{2k-1}) a). \quad (54)$$

In the proof, we will use the following lemma.

Lemma 5.6. *Suppose a is B -valued R-diagonal. Fix $n \geq 1$ and suppose that for all $k \in \{1, \dots, n-1\}$ and all b_1, \dots, b_{2k-1} the equality (53) holds. Then for all $\pi \in \text{NC}(2n) \setminus \{1_{2n}\}$ and all $b_1, \dots, b_{2n-1} \in B$, we have*

$$\begin{aligned} \hat{\alpha}_{(2,1,\dots,2,1)}(\pi)[b_1, \theta(b_2), b_3, \dots, \theta(b_{2n-2}), b_{2n-1}] \\ = \theta(\hat{\alpha}_{(1,2,\dots,1,2)}(\pi)[\theta(b_1), b_2, \theta(b_3), \dots, b_{2n-2}, \theta(b_{2n-1})]). \end{aligned} \quad (55)$$

Proof. This follows by straightforward computation using the recursive formula (5) for the maps $\hat{\alpha}_j(\pi)$ by induction on $|\pi|$, namely, on the number of blocks in π . Suppose $|\pi| = 2$. Then both sides of (55) vanish unless both blocks are of even length. If both blocks are intervals, then $\pi = \{\{1, \dots, 2p\}, \{2p+1, \dots, 2n\}\}$ for some $1 \leq p < n$ and using (53) we have

$$\begin{aligned}
& \hat{\alpha}_{(2,1,\dots,2,1)}(\pi)[b_1, \theta(b_2), b_3, \dots, \theta(b_{2n-2}), b_{2n-1}] \\
&= \alpha_{(2,1,\dots,2,1)}(b_1, \theta(b_2), b_3, \dots, \theta(b_{2p-2}), b_{2p-1})\theta(b_{2p}) \\
&\quad \cdot \alpha_{(2,1,\dots,2,1)}(b_{2p+1}, \theta(b_{2p+2}), b_{2p+3}, \dots, \theta(b_{2n-2}), b_{2n-1}) \\
&= \theta\left(\alpha_{(1,2,\dots,1,2)}(\theta(b_1), b_2, \theta(b_3), \dots, b_{2p-2}, \theta(b_{2p-1}))b_{2p}\right. \\
&\quad \left. \cdot \alpha_{(1,2,\dots,1,2)}(\theta(b_{2p+1}), b_{2p+2}, \theta(b_{2p+3}), \dots, b_{2n-2}, \theta(b_{2n-1}))\right) \\
&= \theta(\hat{\alpha}_{(1,2,\dots,1,2)}(\pi)[\theta(b_1), b_2, \theta(b_3), \dots, b_{2n-2}, \theta(b_{2n-1})]).
\end{aligned}$$

If $|\pi| = 2$ and π has an internal interval of the form $\{2p+1, 2p+2, \dots, 2p+2q\}$ for some $1 \leq p < p+q < n$, then again using (53) we have

$$\hat{\alpha}_{(2,1,\dots,2,1)}(\pi)[b_1, \theta(b_2), b_3, \dots, \theta(b_{2n-2}), b_{2n-1}] \quad (56)$$

$$\begin{aligned}
&= \alpha_{(2,1,\dots,2,1)}\left(b_1, \theta(b_2), b_3, \dots, \theta(b_{2p-2}), b_{2p-1}, \right. \\
&\quad \left. \theta(b_{2p})\alpha_{(2,1,\dots,2,1)}(b_{2p+1}, \theta(b_{2p+2}), \dots, b_{2p+2q-1})\theta(b_{2p+2q}), \right. \\
&\quad \left. b_{2p+2q+1}, \theta(b_{2p+2p+2}), b_{2p+2q+3}, \dots, \theta(b_{2n-2}), b_{2n-1}\right)
\end{aligned} \quad (57)$$

$$\begin{aligned}
&= \alpha_{(2,1,\dots,2,1)}\left(b_1, \theta(b_2), b_3, \dots, \theta(b_{2p-2}), b_{2p-1}, \right. \\
&\quad \left. \theta(b_{2p})\alpha_{(1,2,\dots,1,2)}(\theta(b_{2p+1}), b_{2p+2}, \dots, \theta(b_{2p+2q-1}))b_{2p+2q}, \right. \\
&\quad \left. b_{2p+2q+1}, \theta(b_{2p+2p+2}), b_{2p+2q+3}, \dots, \theta(b_{2n-2}), b_{2n-1}\right)
\end{aligned} \quad (58)$$

$$\begin{aligned}
&= \theta\left(\alpha_{(1,2,\dots,1,2)}\left(\theta(b_1), b_2, \theta(b_3), \dots, b_{2p-2}, \theta(b_{2p-1}), \right. \right. \\
&\quad \left. \left. b_{2p}\alpha_{(1,2,\dots,1,2)}(\theta(b_{2p+1}), b_{2p+2}, \dots, \theta(b_{2p+2q-1}))b_{2p+2q}, \right. \right. \\
&\quad \left. \left. \theta(b_{2p+2q+1}), b_{2p+2p+2}, \theta(b_{2p+2q+3}), \dots, b_{2n-2}, \theta(b_{2n-1})\right)\right)
\end{aligned} \quad (59)$$

$$= \theta(\hat{\alpha}_{(1,2,\dots,1,2)}(\pi)[\theta(b_1), b_2, \theta(b_3), \dots, b_{2n-2}, \theta(b_{2n-1})]). \quad (60)$$

If $|\pi| = 2$ and π has an internal interval of the form $\{2p, 2p+1, \dots, 2p+2q-1\}$ for some $1 \leq p < p+q \leq n$, then again using (53) we have

$$\hat{\alpha}_{(2,1,\dots,2,1)}(\pi)[b_1, \theta(b_2), b_3, \dots, \theta(b_{2n-2}), b_{2n-1}] \quad (61)$$

$$\begin{aligned}
&= \alpha_{(2,1,\dots,2,1)}\left(b_1, \theta(b_2), \dots, b_{2p-3}, \theta(b_{2p-2}), \right. \\
&\quad \left. b_{2p-1}\alpha_{(1,2,\dots,1,2)}(\theta(b_{2p}), b_{2p+1}, \dots, \theta(b_{2p+2q-2}))b_{2p+2q-1}, \right. \\
&\quad \left. \theta(b_{2p+2q}), b_{2p+2p+1}, \dots, \theta(b_{2n-2}), b_{2n-1}\right)
\end{aligned} \quad (62)$$

$$= \alpha_{(2,1,\dots,2,1)} \left(b_1, \theta(b_2), \dots, b_{2p-3}, \theta(b_{2p-2}), \right. \quad (63)$$

$$\left. \theta^{-1} \left(\theta(b_{2p-1}) \alpha_{(2,1,\dots,2,1)} (b_{2p}, \theta(b_{2p+1}), \dots, b_{2p+2q-2}) \theta(b_{2p+2q-1}) \right), \right. \\ \left. \theta(b_{2p+2q}), b_{2p+2p+1}, \dots, \theta(b_{2n-2}), b_{2n-1} \right) \\ = \theta \left(\alpha_{(1,2,\dots,1,2)} \left(\theta(b_1), b_2, \dots, \theta(b_{2p-3}), b_{2p-2}, \right. \quad (64)$$

$$\left. \theta(b_{2p-1}) \alpha_{(2,1,\dots,2,1)} (b_{2p}, \theta(b_{2p+1}), \dots, b_{2p+2q-2}) \theta(b_{2p+2q-1}), \right. \\ \left. b_{2p+2q}, \theta(b_{2p+2p+1}), \dots, b_{2n-2}, \theta(b_{2n-1}) \right) \\ = \theta \left(\hat{\alpha}_{(1,2,\dots,1,2)}(\pi) [\theta(b_1), b_2, \theta(b_3), \dots, b_{2n-2}, \theta(b_{2n-1})] \right). \quad (65)$$

This finishes the proof of the case $|\pi| = 2$.

The induction step when $|\pi| > 2$ is very similar. We see that both sides of (55) vanish unless π has an internal interval of even length, and then, using the induction hypothesis, one gets recursive formulas like in (56)–(65), except that in (57)–(59) and (62)–(64), each $\alpha_{(1,2,\dots,1,2)}(\dots)$ appearing immediately after an equality sign is replaced by $\alpha_{(1,2,\dots,1,2)}(\pi')[\dots]$ and each $\alpha_{(2,1,\dots,2,1)}(\dots)$ is replaced by $\alpha_{(2,1,\dots,2,1)}(\pi')[\dots]$, where π' is obtained from π by removing the corresponding internal interval and renumbering. \square

Proof of Proposition 5.5. The proof is patterned after the proof of Proposition 5.4. The equivalence (a) \iff (b) follows easily from the moment-cumulant formula (2) and Lemma 5.6, by induction on k , keeping in mind that all of the cumulant maps vanish except the $2k$ -th order ones $\beta_k^{(1)} = \alpha_{(1,2,\dots,1,2)}$ and $\beta_k^{(2)} = \alpha_{(2,1,\dots,2,1)}$, ($k \geq 1$).

We now show (b) \implies (c). Endow $B\langle X \rangle$ with the $*$ -operation that extends the given one on B and satisfies $X^* = X$. Let $\Theta : B\langle X \rangle \rightarrow B$ be the B -valued distribution of an element whose odd order B -valued cumulant maps are all zero and whose B -valued cumulant map of order $2k$ is the map γ_k given by

$$\gamma_k(b_1, \dots, b_{2k-1}) = \beta_k^{(2)}(\theta^{-1}(b_1), b_2, \theta^{-1}(b_3), \dots, b_{2k-2}, \theta^{-1}(b_{2k-1})) \\ = \theta \circ \beta_k^{(1)}(b_1, \theta^{-1}(b_2), b_3, \dots, \theta^{-1}(b_{2k-2}), b_{2k-1}), \quad (66)$$

for all $k \geq 1$, where the second equality above is from (53). Write p for the element X of $B\langle X \rangle$. By Proposition 2.4 and the identity (44), Θ is a self-adjoint map. We claim (letting γ_k be shorthand for $\gamma_{(1,1,\dots,1)}$ with 1 repeated $2k$ times and similarly letting $\hat{\gamma}_k$ be shorthand for $\hat{\gamma}_{(1,1,\dots,1)}$), that for every $k \geq 1$, every $\pi \in \text{NC}(2k)$ and every $b_1, \dots, b_{2k-1} \in B$, we have

$$\hat{\gamma}_k(\pi)[b_1, \dots, b_{2k-1}] = \hat{\alpha}_{(2,1,\dots,2,1)}(\pi) [\theta^{-1}(b_1), b_2, \theta^{-1}(b_3), \dots, b_{2k-2}, \theta^{-1}(b_{2k-1})]. \quad (67)$$

If $\pi = 1_{2k}$ then this holds by definition, and the general case is proved by induction on the number $|\pi|$ of blocks in π . For the induction step, if π has an internal interval block of the form $B = \{2r, 2r+1, \dots, 2r+2s-1\}$, for some integers $1 \leq r < r+s \leq k$, then letting

$\pi' \in \text{NC}(2k - 2s)$ be obtained from π by removing the block B and renumbering, we have

$$\begin{aligned}
& \hat{\gamma}_k(\pi)[b_1, \dots, b_{2k-1}] \\
&= \hat{\gamma}_{k-s}(\pi')[b_1, \dots, b_{2r-2}, b_{2r-1}\gamma_s(b_{2r}, \dots, b_{2r+2s-2})b_{2r+2s-1}, b_{2r+2s}, \dots, b_{2k-1}] \\
&= \hat{\alpha}_{(2,1,\dots,2,1)}(\pi') \left[\theta^{-1}(b_1), b_2, \dots, \theta^{-1}(b_{2r-3}), b_{2r-2}, \right. \\
&\quad \left. \theta^{-1} \left(b_{2r-1} \alpha_{(2,1,\dots,2,1)}(\theta^{-1}(b_{2r}), b_{2r+1}, \dots, \theta^{-1}(b_{2r+2s-2})) b_{2r+2s-1} \right), \right. \\
&\quad \left. b_{2r+2s}, \theta^{-1}(b_{2r+2s+1}), \dots, b_{2k-2}, \theta^{-1}(b_{2k-1}) \right] \\
&= \hat{\alpha}_{(2,1,\dots,2,1)}(\pi') \left[\theta^{-1}(b_1), b_2, \dots, \theta^{-1}(b_{2r-3}), b_{2r-2}, \right. \\
&\quad \left. \theta^{-1}(b_{2r-1}) \alpha_{(1,2,\dots,1,2)}(b_{2r}, \theta^{-1}(b_{2r+1}), \dots, b_{2r+2s-2}) \theta^{-1}(b_{2r+2s-1}), \right. \\
&\quad \left. b_{2r+2s}, \theta^{-1}(b_{2r+2s+1}), \dots, b_{2k-2}, \theta^{-1}(b_{2k-1}) \right] \\
&= \hat{\alpha}_{(2,1,\dots,2,1)}(\pi) [\theta^{-1}(b_1), b_2, \theta^{-1}(b_3), \dots, b_{2k-2}, \theta^{-1}(b_{2k-1})],
\end{aligned}$$

where the first and last equalities are by the recursive formula for $\hat{\gamma}$ (see (5)) the second equality is by the induction hypothesis and (66), and the third equality is from (53). The cases when π has an internal interval block of the form $\{2r+1, \dots, 2r+2s\}$ for some integers $1 \leq r < r+s < k$ or when π has only two blocks, both of them intervals, are treated similarly, to prove (67). From the equality (67), the fact that all odd B -valued moments vanish and the moment-cumulant formula, we deduce the equality (54) for all $k \geq 1$ and all b_1, \dots, b_{2k-1} .

Using a crossed product construction and the amalgamated free product construction, there is a B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ containing elements $p, u \in \tilde{A}$ satisfying conditions (i), (ii) and (iii) of (c). (This should be reasonably clear, but see the proof of Lemma 5.7, below for more details.) By construction, (v) holds. Let $\tilde{a} = up \in \tilde{A}$. By Lemma 3.3, \tilde{a} is B -valued R -diagonal. In order to prove that a and \tilde{a} have the same B -valued $*$ -distribution, using Remark 1.4, it will suffice to show that the equations (48) and (49) always hold. To show (48), we use

$$\tilde{a}^* b_1 \tilde{a} b_2 \tilde{a}^* b_3 \tilde{a} b_4 \cdots \tilde{a}^* b_{2k-1} \tilde{a} = p \theta(b_1) p b_2 p \theta(b_3) p \cdots \cdots b_{2k-2} p \theta(b_{2k-1}) p, \quad (68)$$

and (54). To show (49), we use

$$\tilde{a} b_1 \tilde{a}^* b_2 \tilde{a} b_3 \tilde{a}^* b_4 \cdots \tilde{a} b_{2k-1} \tilde{a}^* = u (p b_1 p \theta(b_2) p b_3 p \cdots \theta(b_{2k-2}) p b_{2k-1} p) u^*. \quad (69)$$

Thus, we get

$$\begin{aligned}
& \tilde{\mathcal{E}}(\tilde{a} b_1 \tilde{a}^* b_2 \tilde{a} b_3 \tilde{a}^* b_4 \cdots \tilde{a} b_{2k-1} \tilde{a}^*) = \tilde{\mathcal{E}}(u \tilde{\mathcal{E}}(p b_1 p \theta(b_2) p b_3 p \cdots \theta(b_{2k-2}) p b_{2k-1} p) u^*) \\
&= \theta^{-1}(\tilde{\mathcal{E}}(p b_1 p \theta(b_2) p b_3 p \cdots \theta(b_{2k-2}) p b_{2k-1} p)) \\
&= \theta^{-1}(\mathcal{E}(a^* \theta^{-1}(b_1) a \theta(b_2) a^* \theta^{-1}(b_3) a \cdots \theta(b_{2k-2}) a^* \theta^{-1}(b_{2k-1}) a)) \\
&= \mathcal{E}(a b_1 a^* b_2 a b_3 a^* \cdots b_{2k-2} a b_{2k-1} a^*)
\end{aligned}$$

where the first equality is due to freeness, the second equality is by (iii) of (c), the third equality is from (54) and the fourth equality is (52). This proves we get (49). Thus, a and $\tilde{a} = up$ have the same $*$ -distribution; namely, (c) holds.

The implication (c) \implies (d) is trivially true. Assuming (d), the equality (54) follows by writing

$$\begin{aligned} \mathcal{E}(a^*\theta^{-1}(b_1)ab_2a^*\theta^{-1}(b_3)a\cdots b_{2k-2}a^*\theta^{-1}(b_{2k-1})a) \\ = \tilde{\mathcal{E}}(pu^*\theta^{-1}(b_1)upb_2pu^*\theta^{-1}(b_3)up\cdots b_{2k-2}pu^*\theta^{-1}(b_{2k-1})up) \\ = \tilde{\mathcal{E}}(pb_1pb_2\cdots pb_{2k-1}p). \end{aligned}$$

We now show (d) \implies (a). That a is B -valued R-diagonal follows from Lemma 3.5. To show (52), let $\tilde{a} = up$. By hypothesis, a and \tilde{a} have the same B -valued $*$ -distribution. Of course, the equalities (68) and (69) hold. Therefore, we have

$$\begin{aligned} \mathcal{E}(a^*b_1a\theta(b_2)a^*b_3a\cdots\theta(b_{2k-2})a^*b_{2k-1}a) &= \tilde{\mathcal{E}}(\tilde{a}^*b_1\tilde{a}\theta(b_2)\tilde{a}^*b_3\tilde{a}\cdots\theta(b_{2k-2})\tilde{a}^*b_{2k-1}\tilde{a}) \\ &= \tilde{\mathcal{E}}(p\theta(b_1)p\theta(b_2)p\theta(b_3)p\cdots\theta(b_{2k-2})p\theta(b_{2k-1})p) \\ &= \theta(\tilde{\mathcal{E}}(u(p\theta(b_1)p\theta(b_2)p\theta(b_3)p\cdots\theta(b_{2k-2})p\theta(b_{2k-1})p)u^*)) \\ &= \theta(\tilde{\mathcal{E}}(\tilde{a}\theta(b_1)\tilde{a}^*b_2\tilde{a}\theta(b_3)p\cdots b_{2k-2}\tilde{a}\theta(b_{2k-1})\tilde{a}^*)) \\ &= \theta(\mathcal{E}(a\theta(b_1)a^*b_2a\theta(b_3)p\cdots b_{2k-2}a\theta(b_{2k-1})a^*)). \end{aligned}$$

where the second equality is (68), the third is by condition (ii) of (c) and the fourth is (69). \square

Lemma 5.7. *In Proposition 5.5, if B is a C^* -algebra and if (A, \mathcal{E}) is a B -valued C^* -noncommutative probability space, then in part (c), $(\tilde{A}, \tilde{\mathcal{E}})$ can be realized as a B -valued C^* -noncommutative probability space.*

Proof. In the proof of Proposition 5.5, the $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ was constructed as an algebraic amalgamated free product

$$(\tilde{A}, \tilde{\mathcal{E}}) = (B \rtimes_{\theta}^{\text{alg}} \mathbf{Z}, \mathcal{F}^{\text{alg}}) *_B (B\langle X \rangle, \Theta). \quad (70)$$

Here $B \rtimes_{\theta}^{\text{alg}} \mathbf{Z}$ is an algebraic crossed product of B by the automorphism θ and \mathcal{F}^{alg} is the canonical conditional expectation, and the unitary u of Proposition 5.5(c) is from the element $1 \in \mathbf{Z}$; recall, Θ is defined abstractly by specifying cumulants, and is the $*$ -distribution of the element we called p . If B is a C^* -algebra, then we can take, instead, the crossed product C^* -algebra $B \rtimes_{\theta} \mathbf{Z}$ and the canonical conditional expectation $\mathcal{F} : B \rtimes_{\theta} \mathbf{Z} \rightarrow B$.

We also want a B -valued C^* -noncommutative probability space to replace $(B\langle X \rangle, \Theta)$. Endow $B\langle X_1, X_2 \rangle$ with the $*$ -operation determined by that of B and by setting $X_1^* = X_2$ and let $\tilde{\Theta} : B\langle X_1, X_2 \rangle \rightarrow B$ denote the $*$ -distribution of the pair (a, a^*) . Since (A, \mathcal{E}) is a B -valued C^* -noncommutative probability space, we have (see, for example, [6]), the Hilbert B -module $E = L^2(A, \mathcal{E})$, on which A acts by left multiplication as bounded, adjointable operators. We may without loss of generality assume A is generated as a C^* -algebra by $B \cup \{a\}$. It will be convenient to identify E with $L^2(B\langle X_1, X_2 \rangle, \tilde{\Theta})$, which is obtained by separation and completion, after endowing $B\langle X_1, X_2 \rangle$ with the inner product $\langle y_1, y_2 \rangle = \tilde{\Theta}(y_1^*y_2)$, on which $B\langle X_1, X_2 \rangle$ acts by left multiplication as bounded, adjointable operators on E . Let π denote this action. Letting $y \mapsto \hat{y}$ denote the defining mapping $B\langle X_1, X_2 \rangle \rightarrow E$, the set $\{\hat{b} \mid b \in B\}$ is a complemented subspace of E isomorphic to the Hilbert B -module B , with a self-adjoint projection $P : E \rightarrow B$. We have the conditional expectation $\mathcal{L}(E) \rightarrow B$ given by $Z \mapsto PZ\hat{1}$, and we have $\tilde{\Theta}(y) = P\pi(y)\hat{1}$ for all $y \in B\langle X_1, X_2 \rangle$.

Consider the closed right B -submodules of E ,

$$E_0 = \overline{\text{span}} \{b_0 X_2 b_1 X_1 b_2 X_2 b_3 X_1 \cdots b_{n-2} X_2 b_{n-1} X_1 b_n \mid n \text{ even}, b_0, \dots, b_n \in B\},$$

$$E_1 = \overline{\text{span}} \{b_0 X_1 b_1 X_2 b_2 X_1 b_3 X_2 \cdots b_{n-2} X_1 b_{n-1} X_2 b_n \mid n \text{ odd}, b_0, \dots, b_n \in B\}.$$

Then B (the image of P) is a submodule of E_0 , the spaces E_0 and E_1 are orthogonal to each other with respect to the B -valued inner product and their sum $E_0 + E_1$, which we write $E_0 \oplus E_1$, is also a closed submodule of E . We have the representation $\sigma : B \rightarrow \mathcal{L}(E_0 \oplus E_1)$ given by

$$\sigma(b) = \pi(b)|_{E_0} \oplus \pi(\theta^{-1}(b))|_{E_1}.$$

We also have the bounded operator Y on $E_0 \oplus E_1$ given by

$$Y(e_0 \oplus e_1) = \pi(X_2)e_1 \oplus \pi(X_1)e_0.$$

Since $\pi(X_1)^* = \pi(X_2)$, we easily see that Y is self-adjoint. Thus, the representation σ of B extends to a representation $\sigma : B\langle X \rangle \rightarrow \mathcal{L}(E_0 \oplus E_1)$ by setting $\sigma(X) = Y$. Of course, $\hat{1} \in B \subset E_0$ and, if we identify B with $\sigma(B) \subset \mathcal{L}(E_0 \oplus E_1)$, then we have the conditional expectation $\mathcal{G} : \mathcal{L}(E_0 \oplus E_1) \rightarrow B$ given by $\mathcal{G}(Z) = PZ\hat{1}$. Now we see that, for every $y \in B\langle X \rangle$, we have $\mathcal{G}(\sigma(y)) = \Theta(y)$. Indeed, for a monomial $m = b_0 X b_1 X \cdots b_{n-1} X b_n$, if n is odd, then $m\hat{1} \in E_1$ and $\mathcal{G}(\sigma(m)) = 0 = \Theta(m)$, while for n even, we have

$$\sigma(m)\hat{1} = \pi(b_0 X_2 \theta^{-1}(b_1) X_1 b_2 X_2 \theta^{-1}(b_3) X_1 \cdots b_{n-2} X_2 \theta^{-1}(b_{n-1}) X_1 b_n)\hat{1}$$

and

$$\begin{aligned} \mathcal{G}(\sigma(m)) &= \tilde{\Theta}(b_0 X_2 \theta^{-1}(b_1) X_1 b_2 X_2 \theta^{-1}(b_3) X_1 \cdots b_{n-2} X_2 \theta^{-1}(b_{n-1}) X_1 b_n) \\ &= b_0 \mathcal{E}(a^* \theta^{-1}(b_1) a b_2 a^* \theta^{-1}(b_3) \cdots b_{n-2} a^* \theta^{-1}(b_{n-1}) a) b_n \\ &= b_0 \tilde{\mathcal{E}}(p b_1 p b_2 \cdots p b_{n-1} p) b_n = \Theta(m), \end{aligned}$$

where for the third equality we used (54).

Thus, we may replace $(B\langle X \rangle, \Theta)$ with the B -valued C^* -noncommutative probability space $(\mathcal{L}(E_0 \oplus E_1), \mathcal{G})$, and instead of the algebraic amalgamated free product (70), we let

$$(\tilde{A}, \tilde{\mathcal{E}}) = (B \rtimes_{\theta} \mathbf{Z}, \mathcal{F}) *_B (\mathcal{L}(E_0 \oplus E_1), \mathcal{G}).$$

Identifying p with $\sigma(X) \in \mathcal{L}(E_0 \oplus E_1)$ and $u \in B \otimes_{\theta} \mathbf{Z}$ with the unitary corresponding to the group element 1 in the crossed product construction, we are done. \square

Theorem 5.8. *Let B be a C^* -algebra and suppose (A, \mathcal{E}) is a B -valued C^* -noncommutative probability space. Suppose $a \in A$ and let θ be a $*$ -automorphism of B . Then the following are equivalent:*

(a) *a is B -valued R -diagonal and for all $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$, we have*

$$\begin{aligned} \mathcal{E}(a^* b_1 a \theta(b_2) a^* b_3 a \cdots \theta(b_{2k-2}) a^* b_{2k-1} a) \\ = \theta(\mathcal{E}(a \theta(b_1) a^* b_2 a \theta(b_3) a^* \cdots b_{2k-2} a \theta(b_{2k-1}) a^*)). \end{aligned} \quad (71)$$

(b) *a is B -valued R -diagonal and for all $k \geq 1$, the $2k$ -th order B -valued cumulant maps $\beta_k^{(1)}$ and $\beta_k^{(2)}$ (notation defined near (43)) satisfy, for all $b_1, \dots, b_{2k-1} \in B$,*

$$\beta_k^{(2)}(b_1, \theta(b_2), b_3, \dots, \theta(b_{2k-2}), b_{2k-1}) = \theta(\beta_k^{(1)}(\theta(b_1), b_2, \theta(b_3), \dots, b_{2k-2}, \theta(b_{2k-1}))) \quad (72)$$

(c) *there exists a B -valued C^* -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ with elements $u, p \in \tilde{A}$ satisfying*

(i) *u is a Haar unitary that normalizes B and satisfies $u^* b u = \theta(b)$ for all $b \in B$*

- (ii) $p \geq 0$,
- (iii) $\{u^*, u\}$ and $\{p\}$ are free (over B) with respect to $\tilde{\mathcal{E}}$
- (iv) a and up have the same B -valued $*$ -distribution.

Moreover, the even moments of p in (c) are given by

$$\tilde{\mathcal{E}}(pb_1pb_2 \cdots pb_{2k-1}p) = \mathcal{E}(a^*\theta^{-1}(b_1)ab_2a^*\theta^{-1}(b_3)a \cdots b_{2k-2}a^*\theta^{-1}(b_{2k-1})a). \quad (73)$$

for all $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$.

Proof. The equivalence of (a) and (b), as well as the implication (c) \implies (a) follow from Proposition 5.5. It remains to show (a) \implies (c).

Assuming (a), by Proposition 5.5 and Lemma 5.7, there is a B -valued C^* -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$, and elements u and $p = p^*$ of \tilde{A} such that (i), (iii) and (iv) of (c) hold, the even moments of p are given by (73) and the odd moments of p all vanish. By modifying $(\tilde{A}, \tilde{\mathcal{E}})$, if necessary, we may assume that $p = s|p|$, where s is a symmetry (i.e., a self-adjoint unitary element) that commutes with $|p|$ and with every element of B , satisfies $\tilde{\mathcal{E}}(s) = 0$ and such that $\{s, p\}$ is free from $\{u, u^*\}$. Indeed, we may without loss of generality assume \tilde{A} is the C^* -algebra generated by $\{u, p\}$ and the GNS representation of $\tilde{\mathcal{E}}$ is faithful, in which case \tilde{A} is the amalgamated free product of $C^*(\{u\} \cup B)$ and $C^*(\{p\} \cup B)$ over B . We may enlarge $(\tilde{A}, \tilde{\mathcal{E}})$ to be the amalgamated free product of $C^*(\{u\} \cup B)$ and $C^*(\{p\} \cup B) \otimes (\mathbf{C} \oplus \mathbf{C})$, where the conditional expectation of the latter onto B is $\tilde{\mathcal{E}}|_{C^*(\{p\} \cup B)} \otimes \tau$, where $\tau : \mathbf{C} \oplus \mathbf{C} \rightarrow \mathbf{C}$ is the state sending $1 \oplus -1$ to 0. We let

$$s = 1 \otimes (1 \oplus -1) \in C^*(\{p\} \cup B) \otimes (\mathbf{C} \oplus \mathbf{C})$$

and note that the pair (u, p) has the same B -valued $*$ -distribution as $(u, s|p|)$. Thus, a has the same $*$ -distribution as $us|p|$. We observe that, with respect to $\tilde{\mathcal{E}}$, us is a B -normalizing Haar unitary with $(us)^*b(us) = s\theta(b)s = \theta(b)$.

We need only show that $\{us, su^*\}$ is free from $\{|p|\}$ with respect to $\tilde{\mathcal{E}}$. This is straightforward to verify. Indeed, we need only show that every alternating product in the sets $\{(us)^k \mid k \geq 1\} \cup \{(su^*)^k \mid k \geq 1\}$ and $\{|p|^k - \tilde{\mathcal{E}}(|p|^k) \mid k \geq 1\}$ evaluates to zero under $\tilde{\mathcal{E}}$. However, keeping in mind that s commutes with $|p|$, we may rewrite each such product as an alternating product in the sets $\{u, u^*\}$ and

$$\{|p|^k - \tilde{\mathcal{E}}(|p|^k) \mid k \geq 1\} \cup \{s(|p|^k - \tilde{\mathcal{E}}(|p|^k)) \mid k \geq 1\},$$

and such an alternating product evaluates to 0 under $\tilde{\mathcal{E}}$ by freeness of $\{|p|, s\}$ and $\{u, u^*\}$. \square

In the case of $B = \mathbf{C}$, Theorem 5.8 reduces to the following, which is certainly well known, though we didn't find a good reference. It would follow straightforwardly from Proposition 2.6 of [7].

Corollary 5.9. *Let (A, τ) be a tracial (scalar-valued) C^* -noncommutative probability space and let $a \in A$. Then the following are equivalent:*

- (a) a is R -diagonal.
- (b) *There exists a tracial (scalar-valued) C^* -noncommutative probability space $(\tilde{A}, \tilde{\tau})$ and elements $u, p \in \tilde{A}$ such that*
 - (i) u is a Haar unitary,
 - (ii) $p \geq 0$,
 - (iii) $\{u^*, u\}$ and $\{p\}$ are free with respect to $\tilde{\tau}$
 - (iv) a and up have the same $*$ -distribution.

Proof. We apply Theorem 5.8 in the case $B = \mathbf{C}$. Then of course each cumulant map $\beta_k^{(i)}$ is just a real number and θ is the identity map. The condition (72) just becomes $\beta_k^{(2)} = \beta_k^{(1)}$ for every k , and this holds, by Proposition 5.1, because we assume τ is a trace. Now the equivalence of (a) and (b) above follows from the equivalence of (b) and (c) in Theorem 5.8. \square

For an element a in a B -valued W^* -noncommutative probability space whose polar decomposition is $a = u|a|$, the joint B -valued $*$ -distribution of $\{u, |a|\}$ is completely determined by the B -valued $*$ -distribution of a , and vice-versa. Thus, the following corollary follows from the Theorem 5.8.

Corollary 5.10. *Let B be a von Neumann algebra and suppose (A, \mathcal{E}) is a B -valued W^* -noncommutative probability space whose GNS-representation is faithful. Suppose $a \in A$ has zero kernel and dense range and let θ be a normal $*$ -automorphism of B . Then the following are equivalent:*

- (a) *a is B -valued R -diagonal and for all $k \geq 1$ and all $b_1, \dots, b_{2k-1} \in B$, (71) holds.*
- (b) *a is B -valued R -diagonal and for all $k \geq 1$, the $2k$ -th order B -valued cumulant maps $\beta_k^{(1)}$ and $\beta_k^{(2)}$ satisfy (72), for all $b_1, \dots, b_{2k-1} \in B$.*
- (c) *Letting $a = u|a|$ be the polar decomposition of a ,*
 - (i) *u is a Haar unitary that normalizes B and satisfies $u^*bu = \theta(b)$ for all $b \in B$,*
 - (ii) *$\{u^*, u\}$ and $\{|a|\}$ are free (over B) with respect to $\tilde{\mathcal{E}}$.*

Example 5.11. Perhaps the easiest example of a B -valued R -diagonal element satisfying the conditions of Theorem 5.8 is when $p \in B$. In this case, the operator a can be realized in the crossed product C^* -algebra $B \rtimes_{\theta} \mathbf{Z}$ with respect to the canonical conditional expectation onto B . In the case when B is a commutative von Neumann algebra with a specified normal, faithful tracial state and θ is a trace-preserving normal automorphism that is ergodic, such B -valued R -diagonal operators were studied in [5] and, among other results, their Brown measures were computed.

6. ALGEBRA-VALUED CIRCULAR ELEMENTS

In this section, we examine algebra-valued circular elements, which are a very special class of algebra-valued R -diagonal elements.

As before, let B be a unital $*$ -algebra and let (A, \mathcal{E}) be a B -valued $*$ -noncommutative probability space; let $a \in A$; for convenience we label $a_1 = a$ and $a_2 = a^*$ and use the involution $s : \{1, 2\} \rightarrow \{1, 2\}$ with $s(1) = 2$. Let $\Theta : B\langle X_1, X_2 \rangle \rightarrow B$ be the B -valued $*$ -distribution of (a_1, a_2) , where we set $X_1^* = X_2$. Let $J = \bigcup_{n \geq 1} \{1, 2\}^n$ and let $(\alpha_j)_{j \in J}$ be the family of cumulant maps for the pair (a_1, a_2) .

Definition 6.1. We say that a is *B -valued circular* if $\alpha_j = 0$ whenever $j \in J$ and $j \notin \{(1, 2), (2, 1)\}$.

Clearly, B -valued circular elements are B -valued R -diagonal elements. The notion of B -valued circular first appeared in [11], where Śniady proved interesting combinatorial results about a certain B -valued circular operator for $B = L^\infty[0, 1]$ that he noted was equal to the quasinilpotent DT-operator T from [3] (and a proof that it is, in fact, B -valued circular can be found in [4]) — see Example 6.7 for more details. The notion of B -valued circular operators is closely connected to that of B -valued semicircular (also called B -Gaussian) operators that were introduced by Speicher [13] and were studied and constructed on Fock spaces by Shlyakhtenko [10].

At the purely algebraic level, consider a family $(x_i)_{i \in I}$ of B -valued random variables in a B -valued noncommutative probability space. Let $J = \bigcup_{n \geq 1} I^n$ and denote by $(\gamma_j)_{j \in J}$ the B -valued cumulant maps associated to this family. The family is said to be *centered B -valued semicircular* if $\gamma_j = 0$ for every j of length not equal to 2.

In fact, the following result is immediate from the definitions:

Proposition 6.2. *Suppose a is an element of a B -valued $*$ -noncommutative probability space and let*

$$x_1 = \operatorname{Re} a = \frac{a + a^*}{2}, \quad x_2 = \operatorname{Im} a = \frac{a - a^*}{2i}.$$

Let $J = \bigcup_{n \geq 1} \{1, 2\}^n$ and let $(\alpha_j)_{j \in J}$ be the family of cumulant maps for the pair (a_1, a_2) . Let $(\gamma_j)_{j \in J}$ be the family of cumulant maps for the pair (x_1, x_2) . Then the following are equivalent:

- (i) *a is B -valued circular*
- (ii) *the pair (x_1, x_2) is centered B -valued semicircular, $\gamma_{(1,1)} = \gamma_{(2,2)}$ and $\gamma_{(1,2)} = -\gamma_{(2,1)}$.*

Furthermore, when the above conditions hold, we have

$$\gamma_{(1,1)} = \frac{1}{4} (\alpha_{(1,2)} + \alpha_{(2,1)}), \quad \gamma_{(1,2)} = \frac{i}{4} (\alpha_{(1,2)} - \alpha_{(2,1)}).$$

Using Speicher's result that freeness is equivalent to vanishing of mixed cumulants (see section 3.3 of [13]), we have:

Corollary 6.3. *If a is a B -valued circular element with associated cumulant maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$, then $\operatorname{Re} a$ and $\operatorname{Im} a$ are free with respect to the conditional expectation onto B if and only if $\alpha_{(1,2)} = \alpha_{(2,1)}$*

Suppose B is a C^* -algebra and Θ is the B -valued distribution of a family of B -valued random variables indexed by I . We say that Θ is positive if $\Theta(p^*p) \geq 0$ for every $p \in B\langle X_i \mid i \in I \rangle$. In general, it can be difficult to decide whether Θ is positive knowing only the B -valued cumulant maps of the family. However, in the case that the family is B -valued semicircular, an answer is provided by Theorem 4.3.1 of [13]. In the case that I is finite, this condition is equivalent to complete positivity of the covariance $\eta : B \rightarrow M_{|I|}(B)$, which is defined by

$$\eta(b) = (\gamma_{(i_1, i_2)}(b))_{i_1, i_2 \in I}.$$

In the case of the family (x_1, x_2) of real and imaginary parts of a B -valued circular system from Proposition 6.2, the covariance $\eta : B \rightarrow M_2(B) = M_2(\mathbf{C}) \otimes B$ is given by

$$\eta = \frac{1}{2} \left(\frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \otimes \alpha_{(1,2)} + \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \otimes \alpha_{(2,1)} \right).$$

Thus, we have the following corollary of Speicher's theorem mentioned above and of Shlyakhtenko's construction [10] of B -valued semicircular elements.

To recall the set-up, let B be a C^* -algebra as before, consider a B -valued circular element a , let $(a_1, a_2) = (a, a^*)$ and let Θ be the distribution and $\alpha_{(1,2)}, \alpha_{(2,1)}$ the cumulant maps of (a_1, a_2) , as usual. Endow the algebra $B\langle X_1, X_2 \rangle$ with the $*$ -operation that extends the given one on B by setting $X_1^* = X_2$.

Corollary 6.4. *The distribution Θ is positive if and only if $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ are completely positive maps from B into itself. In this case, Θ is the $*$ -distribution of a B -valued circular element in a B -valued C^* -noncommutative probability space.*

Turning back to the case of B a general $*$ -algebra, without assuming anything about boundedness or positivity, from Proposition 5.1, we immediately see the following:

Proposition 6.5. *If a is a B -valued circular element with associated cumulant maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ and if τ is a tracial linear functional on B , then $\tau \circ \mathcal{E}$ is tracial on $\text{alg}(B \cup \{a, a^*\})$ if and only if, for all $b_1, b_2 \in B$, we have*

$$\tau(\alpha_{(1,2)}(b_1)b_2) = \tau(b_1\alpha_{(2,1)}(b_2)).$$

We will now examine the power series considered in Section 4 for a B -valued circular element a . Theorem 4.2 yields

Proposition 6.6. *Consider the formal power series (in the sense described in Section 4)*

$$F(b_1, b_2) = \sum_{n=0}^{\infty} \mathcal{E}((ab_1a^*b_2)^n) \quad (74)$$

$$G(b_1, b_2) = \sum_{n=0}^{\infty} \mathcal{E}((a^*b_1ab_2)^n). \quad (75)$$

for a B -valued circular element a . Then

$$F(b_1, b_2) = 1 + \alpha_{(1,2)}(b_1G(b_2, b_1))b_2F(b_1, b_2), \quad (76)$$

$$G(b_1, b_2) = 1 + \alpha_{(2,1)}(b_1F(b_2, b_1))b_2G(b_1, b_2). \quad (77)$$

In the case of a B -valued C^* -noncommutative probability space, the series (74)–(75) define B -valued holomorphic functions with domain equal to the subset of $B \oplus B$ consisting of all pairs (b_1, b_2) such that $\|b_1\| \|b_2\| < \|a\|^2$. From these functions one can recover only some of the information about the B -valued distribution of a . Later, in the appendix, we will study a particular case of these functions evaluated at $b_1, b_2 \in \mathbf{C}1$.

Example 6.7. As observed by Śniady [11], (see [4] for a proof), the quasinilpotent DT-operator T from [3] is a $L^\infty[0, 1]$ -valued circular operator, with cumulant maps given by

$$\alpha_{(1,2)}(f)(x) = \int_x^1 f(t) dt, \quad \alpha_{(2,1)}(f)(x) = \int_0^x f(t) dt.$$

We note that if θ is the automorphism of $L^\infty[0, 1]$ corresponding to the homeomorphism $t \mapsto 1 - t$ of $[0, 1]$, then

$$\alpha_{(2,1)}(b) = \theta(\alpha_{(1,2)}(\theta(b))).$$

Thus, Corollary 5.10 applies and we have the following result.

Corollary 6.8. *Let T be a quasinilpotent DT-operator in an $L^\infty[0, 1]$ -valued W^* -noncommutative probability space (A, \mathcal{E}) with \mathcal{E} faithful and let $T = U|T|$ be its polar decomposition. Then*

- (i) U is an $L^\infty[0, 1]$ -normalizing Haar unitary element,
- (ii) for every $b \in L^\infty[0, 1]$ we have $U^*bU = \theta(b)$,
- (iii) $\{U^*, U\}$ and $\{|T|\}$ are free with respect to \mathcal{E} .

As advertised, here is an application of Lemma 5.3, yielding a B -valued R-diagonal element a with B two-dimensional and such that in the polar decomposition $a = u|a|$ of a , u is unitary but $\{u, u^*\}$ and $\{|a|\}$ are not free over B .

Example 6.9. Take $B = \mathbf{C} \oplus \mathbf{C}$, $\tau_B(\lambda_1 \oplus \lambda_2) = \frac{\lambda_1 + \lambda_2}{2}$ and define

$$\begin{aligned} \alpha_{(1,2)}(\lambda_1 \oplus \lambda_2) &= \frac{\lambda_1}{2} \oplus \left(\frac{\lambda_1}{2} + \lambda_2 \right) \\ \alpha_{(2,1)}(\lambda_1 \oplus \lambda_2) &= \frac{\lambda_1 + \lambda_2}{2} \oplus \lambda_2. \end{aligned}$$

Then $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ are completely positive and satisfy the condition of Proposition 6.5 for traciality. Thus, by Corollary 6.4, there is a B -valued C^* -noncommutative probability space (A, \mathcal{E}) containing a B -valued circular element a with corresponding cumulant maps $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$; we assume A is generated as a C^* -algebra by $B \cup \{a\}$, and that the GNS representation of \mathcal{E} is faithful; by Proposition 6.5, $\tau = \tau_B \circ \mathcal{E}$ is a positive trace on A ; since it has faithful GNS representation, it is faithful. We also have $\alpha_{(2,1)}(1) = 1$ and $\alpha_{(1,2)}(1) \neq 1$. Thus by Lemma 5.3, a does not have the same B -valued $*$ -distribution as any element up in any B -valued $*$ -noncommutative probability space with u unitary, with p self-adjoint and with $\{u, u^*\}$ and $\{p\}$ free over B . In particular, we may take the von Neumann algebra generated by the image of the GNS representation of τ we get a larger B -valued $*$ -noncommutative probability space $(\tilde{A}, \tilde{\mathcal{E}})$ and in \tilde{A} we have the polar decomposition $a = u|a|$ of a . By Proposition A.1 below, a has zero kernel, so u is unitary. By Proposition 5.2, u is a Haar unitary. But $\{u, u^*\}$ and $\{|a|\}$ are not free with respect to $\tilde{\mathcal{E}}$.

APPENDIX A. ON A DISTRIBUTION

In this appendix, we describe investigations of the distribution $\mu_{a^*a} = \mu_{aa^*}$ of the element aa^* with respect to the trace τ , where a is the B -valued circular element described in Example 6.9. A Mathematica Notebook file containing the detailed calculations will be made available with this paper.

Using the equations (74) and (75) with $b_1 = b_2 = s1$ for s in some neighborhood of 0 in \mathbf{C} and letting $z = s^2$, we have

$$\begin{aligned} f(z) &:= F(s1, s1) = \sum_{n=0}^{\infty} \mathcal{E}((aa^*)^n) z^n \\ g(z) &:= G(s1, s1) = \sum_{n=0}^{\infty} \mathcal{E}((a^*a)^n) z^n. \end{aligned}$$

We are interested in the moment generating function for aa^* with respect to the trace τ , and this is the function

$$h(z) := \tau(f(z)) = \sum_{n=0}^{\infty} \tau((aa^*)^n) z^n = \sum_{n=0}^{\infty} \tau((a^*a)^n) z^n = \tau(g(z)).$$

The recursive relations in Proposition 6.6 yield

$$\begin{aligned} f(z) &= 1 + z \alpha_{(1,2)}(g(z)) f(z) \\ g(z) &= 1 + z \alpha_{(2,1)}(f(z)) g(z). \end{aligned}$$

Since $B = \mathbf{C} \oplus \mathbf{C}$ and f and g are B -valued functions, we write $f = f_1 \oplus f_2$ and $g = g_1 \oplus g_2$. Using the definitions of $\alpha_{(1,2)}$ and $\alpha_{(2,1)}$ from Example 6.9, we have the recursive relations

$$f_1 = 1 + z \left(\frac{g_1}{2} \right) f_1 \tag{78}$$

$$f_2 = 1 + z \left(\frac{g_1}{2} + g_2 \right) f_2 \tag{79}$$

$$g_1 = 1 + z \left(\frac{f_1 + f_2}{2} \right) g_1 \tag{80}$$

$$g_2 = 1 + z f_2 g_2. \tag{81}$$

We substitute $h = (f_1 + f_2)/2$ and, using simple elimination, arrive at a polynomial identity for h :

$$8z^3h^4 - 20z^2h^3 + 8z(z+2)h^2 + (z^2 - 12z - 4)h + 4 = 0. \quad (82)$$

This allows computation of arbitrarily many terms of the series expansion for h around 0, and we find

$$h(z) = 1 + z + \frac{9}{4}z^2 + \frac{13}{2}z^3 + \frac{341}{16}z^4 + \frac{1207}{16}z^5 + \frac{17985}{64}z^6 + O(|z|^7). \quad (83)$$

The Stieltjes transform $G = G_{\mu_{a^*a}}$ of the measure μ_{a^*a} is, for w in the complement of the closed support of μ_{a^*a} ,

$$G(w) = \int_{\mathbf{R}} \frac{1}{w-t} d\mu_{a^*a}(t) = w^{-1}h(w^{-1}).$$

Note that $G(w)$ is real when w is real and $|w|$ is large. From (82) we get the identity

$$8G^4w^2 - 20G^3w^2 + 8G^2w(2w+1) + G(-4w^2 - 12w + 1) + 4w = 0. \quad (84)$$

Stieltjes inversion can be used to find the measure μ_{a^*a} from knowledge of the algebraic function G : μ_{a^*a} has an atom at a point $t_0 \in \mathbf{R}$ if and only if,

$$\liminf_{\epsilon \rightarrow 0^+} \epsilon G(t_0 + \epsilon i) > 0, \quad (85)$$

and then the limit exists and the value of this limit equals $\mu_{a^*a}(\{t_0\})$ while elsewhere on the real line, μ_{a^*a} has density given by

$$\frac{d\mu_{a^*a}}{dt}(t) = \lim_{\epsilon \rightarrow 0^+} \frac{-\operatorname{Im} G(t + i\epsilon)}{\pi}.$$

Note that, since a^*a is positive and bounded, μ_{a^*a} is compactly supported in $[0, \infty)$.

Of course, an explicit formula for G can be found involving radicals, (solving the quartic) and care must be taken to find the correct branch of G near the real axis; the correct branch of G is the one that near $w = \infty$ has asymptotics $G(w) = \frac{1}{w} + O(|w|^{-2})$. However, the following facts can be seen by a somewhat less arduous analysis:

Proposition A.1. *The element a has zero kernel but is not invertible. Its norm is the root of the polynomial*

$$16x^8 - 160x^6 + 540x^4 - 680x^2 + 27$$

that is approximately equal to 2.18942.

Proof. Using the method of Newton's polytope to find Puiseux series asymptotic expansions for the algebraic function roots of the polynomial equation (84), we find that near $w = 0$, the four algebraic functions are

$$G_1(w) = -4w - 48w^2 + O(|w|^3) \quad (86)$$

$$G_j(w) = -\frac{1}{2}w^{-2/3} + \frac{2}{3}w^{-1/3} + \frac{5}{6} + O(|w|^{1/3}) \quad (j = 2, 3, 4) \quad (87)$$

where the three roots G_j for $j = 2, 3, 4$ result from choosing the different branches of $w^{1/3}$. We set G_2 to be the one that makes $w^{1/3}$ negative real when w is negative real. Already we see that the Stieltjes transform G fails the condition (85) at $t_0 = 0$, so $\mu_{aa^*} = \mu_{a^*a}$ has no atom at 0 and, thus, a has zero kernel and zero co-kernel.

Since μ_{a^*a} has no support in $(-\infty, 0)$, we see that the Stieltjes transform must be equal to either G_1 or G_2 near $w = 0$ (and in the domain of G), since as w approaches negative numbers near zero from above, both $\operatorname{Im} G_3$ and $\operatorname{Im} G_4$ approach nonzero numbers. To see

that a is not invertible, it will suffice to see that $G = G_2$, since it will yield nonzero density for μ_{a^*a} on some interval $[0, \delta)$.

The discriminant of the polynomial in (84), with respect to the variable G , is

$$-64w^4(16w^4 - 160w^3 + 540w^2 - 680w + 27), \quad (88)$$

whose real roots, other than 0, are approximately 0.0410263, and 4.79356. The coefficient of G^4 in (84) is w^2 , which has roots only at 0. Thus, the only place on the real axis where G can diverge to ∞ is 0, and the only places where two or more of the four algebraic roots of (84) can agree is where w is one of the roots of the polynomial (88). The polynomial (84) has real coefficients, so roots come in complex conjugate pairs. Thus $G(w)$, which is real and negative for $w \ll 0$, must remain real as w approaches 0 through negative numbers. Moreover, setting G equal to 0 in (84) yields $w = 0$, so G is nonvanishing on the negative real axis. Therefore, $G(w)$ is real and strictly negative for all $w \in (-\infty, 0)$. However, this is clearly not possible for the function G_1 , according to the asymptotics (86). Thus, $G = G_2$ and a is not invertible.

Similar considerations show that the maximum of the support of μ_{a^*a} , which is equal to $\|a\|^2$, can only be one of the roots of (88). However, the coefficients in the moment series (83) yield lower bounds on $\|a\|^2$; already the coefficient of z yields $\|a\|^2 \geq \tau((aa^*)) = 1$. So $\|a\|^2$ equals the largest of the real roots of (88), as required for the assertion in the proposition about $\|a\|$. \square

Remark A.2. From the asymptotics (87), we easily see that the density of the measure μ_{a^*a} behaves asymptotically like

$$\frac{d\mu_{a^*a}}{dt}(t) = \frac{\sqrt{3}}{4\pi}t^{-2/3} + O(t^{-1/3})$$

as t approaches 0 from the right. Thus, the distribution $\mu_{|a|}$ has density whose asymptotic expansion is

$$\frac{d\mu_{|a|}}{ds}(s) = 2s \frac{d\mu_{a^*a}}{dt}(s^2) = \frac{\sqrt{3}}{2\pi}s^{-1/3} + O(s^{1/3})$$

as s tends to 0 from the right. We calculated the density of $\mu_{|a|}$ numerically; a plot is in Figure 2; for comparison purposes, the density $\frac{d\mu_{|z|}}{ds}(s) = \frac{1}{\pi}\sqrt{4-s^2}$ of the quarter-circular element $|z|$ is plotted on the same grid, with the dashed line. For details see the Mathematica Notebook file that is available with this paper.

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FIGURE 2. The density of the measure $\mu_{|a|}$.

